CSE 167:
Introduction to Computer Graphics
Lecture #11: Bezier Curves

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Announcements

- Project 5 due Friday
- Heads Up: CSE 190
  - Advanced Computer Graphics
  - Prof. Ravi Ramamoorthi
  - http://cseweb.ucsd.edu/~ravir/190/2015/190.html

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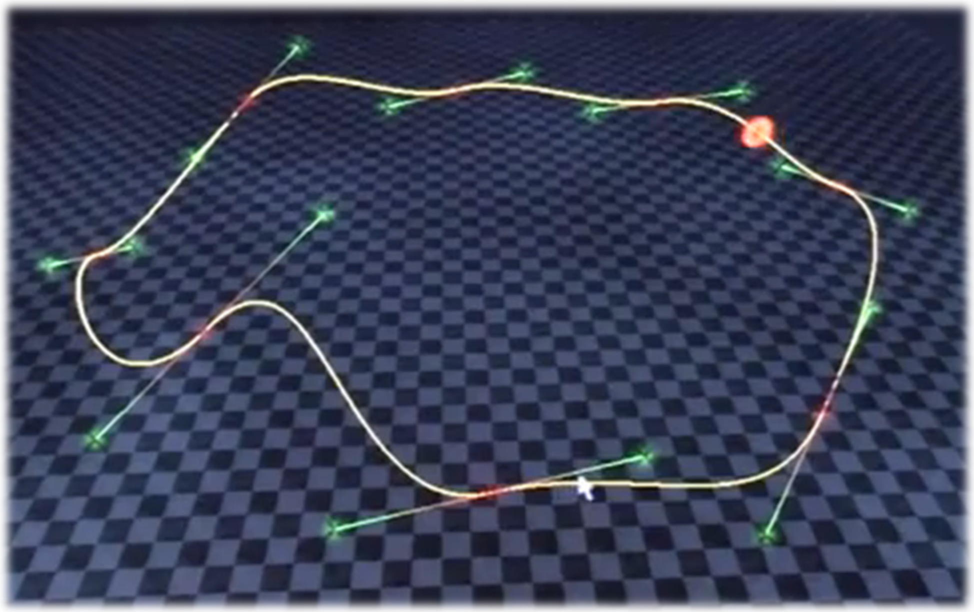
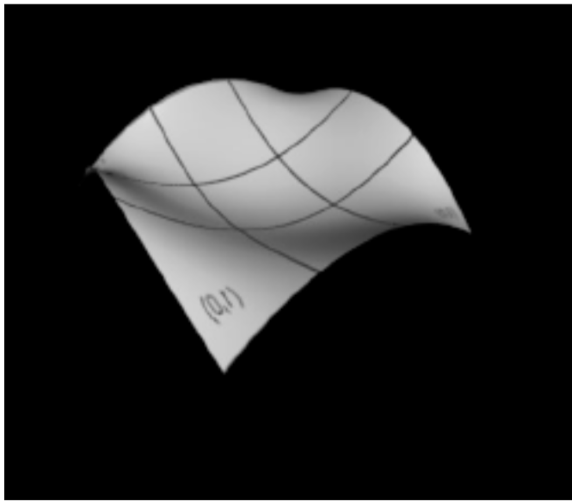
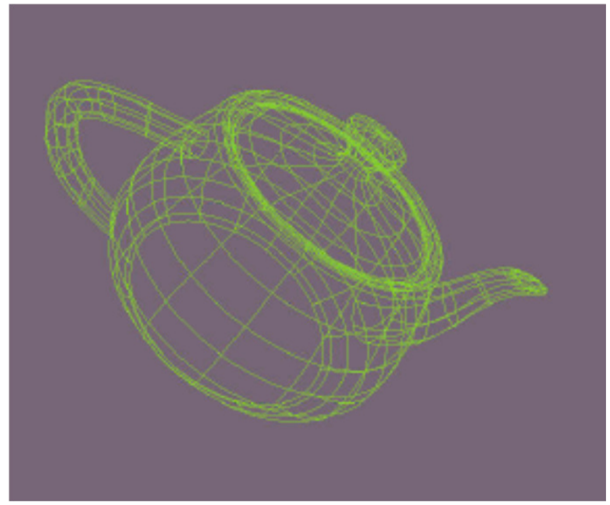
Video

- Bezier Curves
  - http://www.youtube.com/watch?v=hlDYJNEiYvU



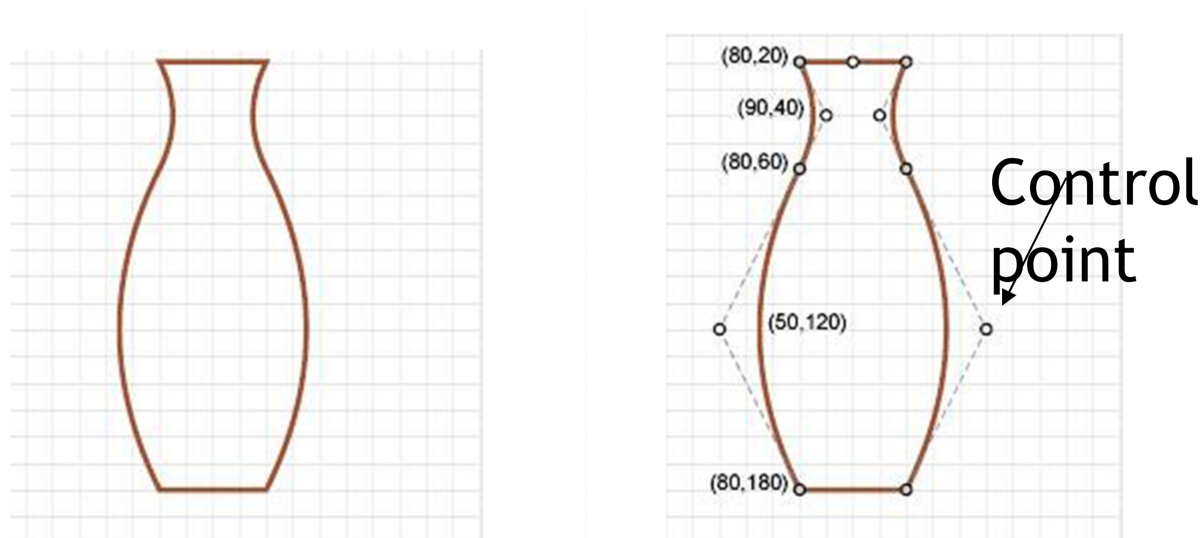
Curves

- Can be generalized to surface patches

Curve Representation

- Specify many points along a curve, connect with lines?
  - Difficult to get precise, smooth results across magnification levels
  - Large storage and CPU requirements
  - How many points are enough?
- Specify a curve using a small number of "control points"
  - Known as a *spline curve* or just *spline*



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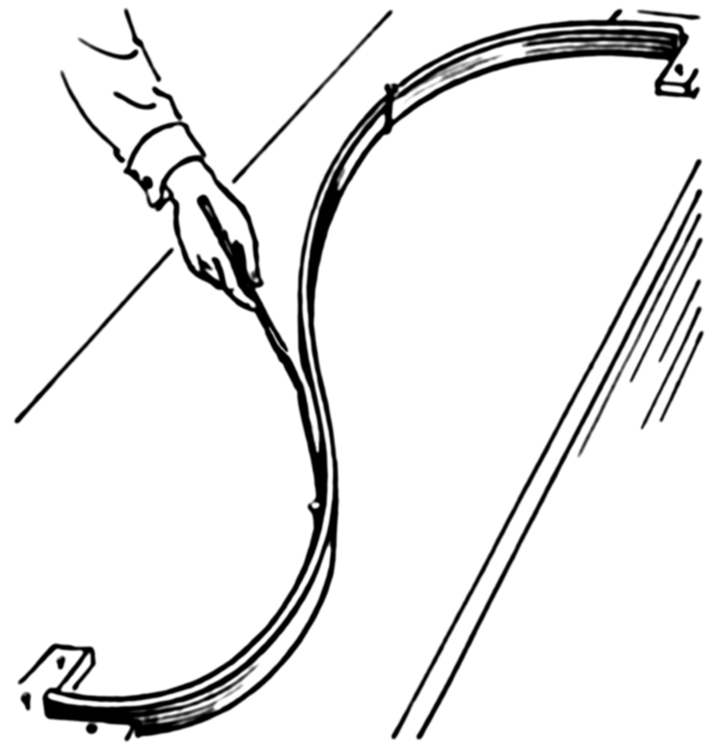
Spline: Definition

- Wikipedia:
  - Term comes from flexible spline devices used by shipbuilders and draftsmen to draw smooth shapes.
  - Spline consists of a long strip fixed in position at a number of points that relaxes to form a smooth curve passing through those points.



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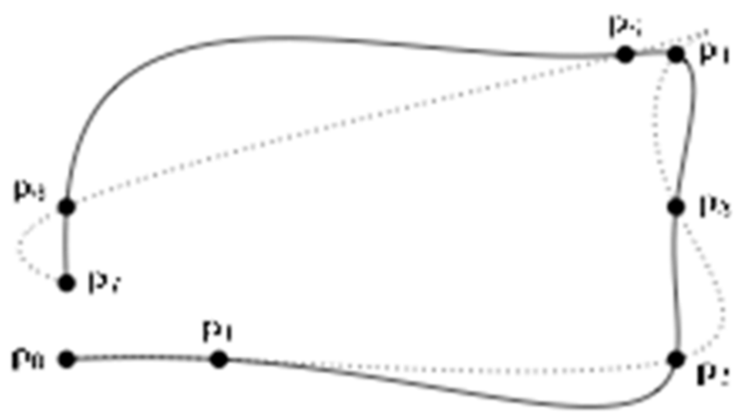
Lecture Overview

- Polynomial Curves
  - Introduction
  - Polynomial functions
- Bézier Curves
  - Introduction
  - Drawing Bézier curves
  - Piecewise Bézier curves

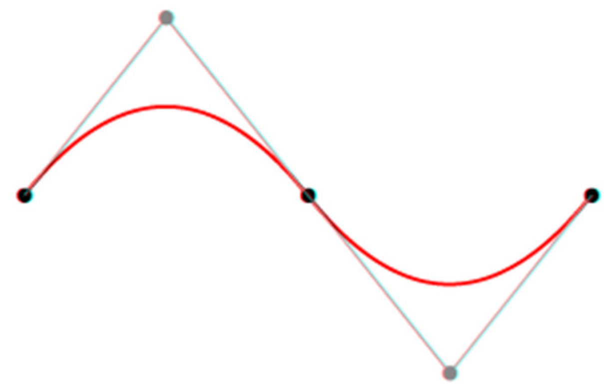
Interpolating Control Points

- "Interpolating" means that curve goes through all control points
- Seems most intuitive
- Surprisingly, not usually the best choice
  - Hard to predict behavior
  - Hard to get aesthetically pleasing curves



Approximating Control Points

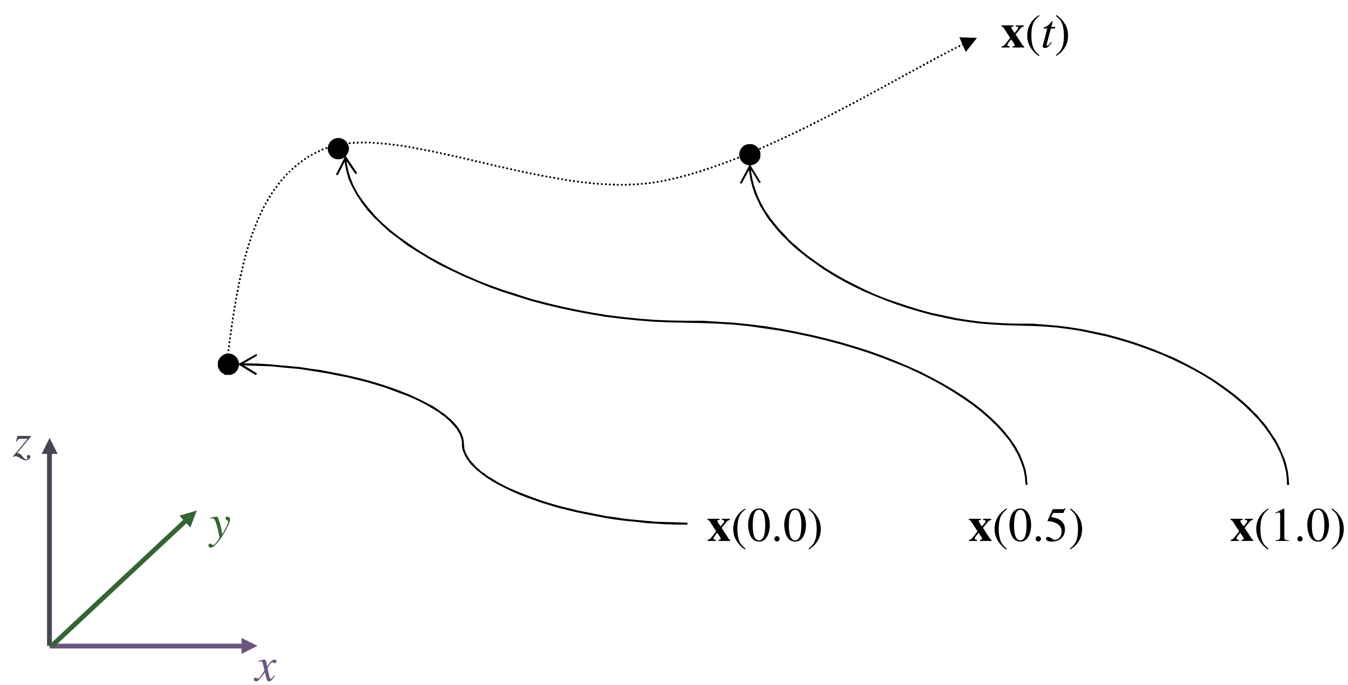
- Curve is "influenced" by control points



- Various types
- Most common: polynomial functions
  - Bézier spline (our focus)
  - B-spline (generalization of Bézier spline)
  - NURBS (Non Uniform Rational Basis Spline): used in CAD tools

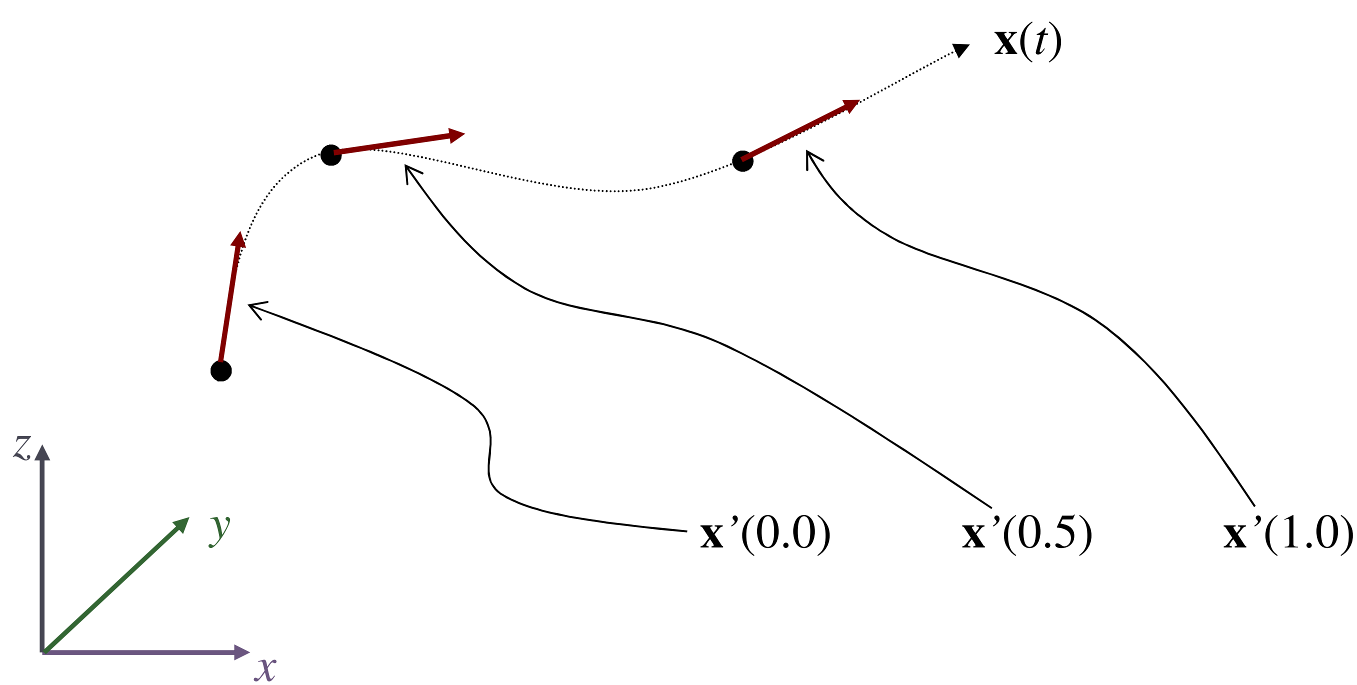
Mathematical Definition

- A vector valued function of one variable $\mathbf{x}(t)$
  - Given t, compute a 3D point $\mathbf{x}=(x,y,z)$
  - Could be interpreted as three functions: $x(t)$, $y(t)$, $z(t)$
  - Parameter t "moves a point along the curve"



Tangent Vector

- Derivative $\mathbf{x}'(t) = \dfrac{d\mathbf{x}}{dt} = (x'(t), y'(t), z'(t))$
- Vector x' points in direction of movement
- Length corresponds to speed



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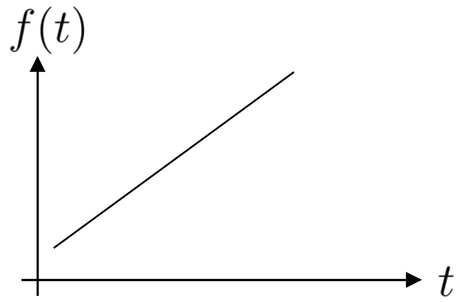
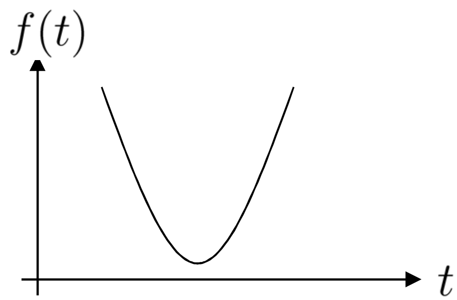
Polynomial Functions

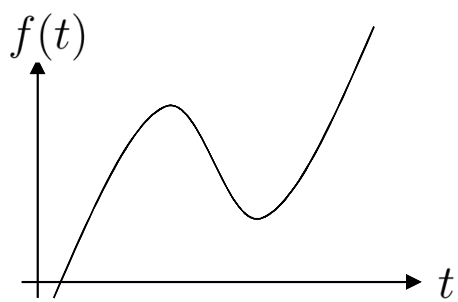
- Linear: (1st order) $$f(t) = at + b$$



- Quadratic: (2nd order) $$f(t) = at^2 + bt + c$$



- Cubic: (3rd order) $$f(t) = at^3 + bt^2 + ct + d$$



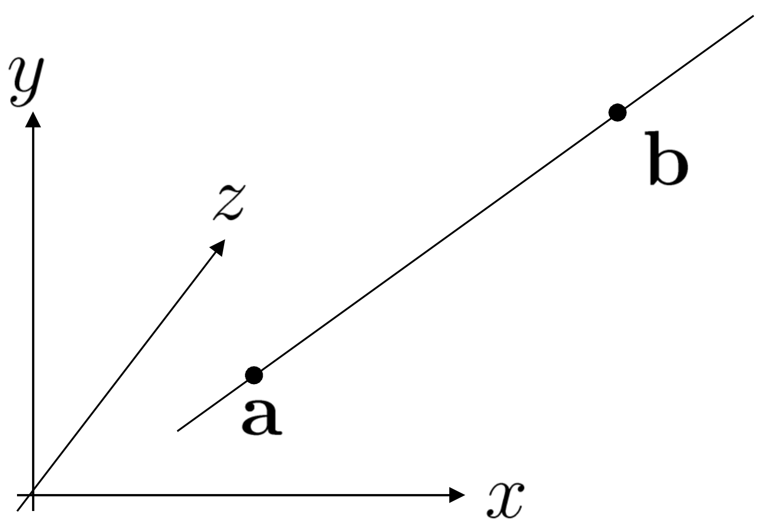
Polynomial Curves

- Linear $\mathbf{x}(t) = \mathbf{a}t + \mathbf{b}$

$$\mathbf{x} = (x, y, z), \mathbf{a} = (a_x, a_y, a_z), \mathbf{b} = (b_x, b_y, b_z)$$

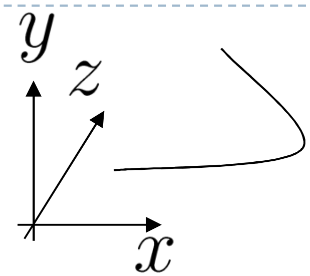
- Evaluated as:

$$x(t) = a_x t + b_x$$
$$y(t) = a_y t + b_y$$
$$z(t) = a_z t + b_z$$

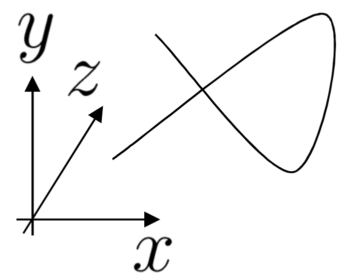


Polynomial Curves

- Quadratic: $\mathbf{x}(t) = \mathbf{a}t^2 + \mathbf{b}t + \mathbf{c}$
  (2nd order)



- Cubic: $\mathbf{x}(t) = \mathbf{a}t^3 + \mathbf{b}t^2 + \mathbf{c}t + \mathbf{d}$
  (3rd order)



- We usually define the curve for $0 \leq t \leq 1$

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Control Points

- Polynomial coefficients $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ can be interpreted as *control points*
  - Remember: $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ have x,y,z components each
- Unfortunately, they do not intuitively describe the shape of the curve
- Goal: intuitive control points

Control Points

- How many control points?
  - Two points define a line (1st order)
  - Three points define a quadratic curve (2nd order)
  - Four points define a cubic curve (3rd order)
  - $k+1$ points define a k-order curve
- Let's start with a line…

First Order Curve

- Based on linear interpolation (LERP)
  - Weighted average between two values
  - "Value" could be a number, vector, color, …
- Interpolate between points $\mathbf{p}_0$ and $\mathbf{p}_1$ with parameter t
  - Defines a "curve" that is straight (first-order spline)
  - $t=0$ corresponds to $\mathbf{p}_0$
  - $t=1$ corresponds to $\mathbf{p}_1$
  - $t=0.5$ corresponds to midpoint

$$\mathbf{x}(t) = Lerp\left(t,\ \mathbf{p}_0,\ \mathbf{p}_1\right) = \left(1-t\right)\mathbf{p}_0 + t\ \mathbf{p}_1$$

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Linear Interpolation

- Three equivalent ways to write it
  - Expose different properties

1. Regroup for points **p**

$$\mathbf{x}(t) = \mathbf{p}_0(1-t) + \mathbf{p}_1 t$$

2. Regroup for t

$$\mathbf{x}(t) = (\mathbf{p}_1 - \mathbf{p}_0)t + \mathbf{p}_0$$

3. Matrix form

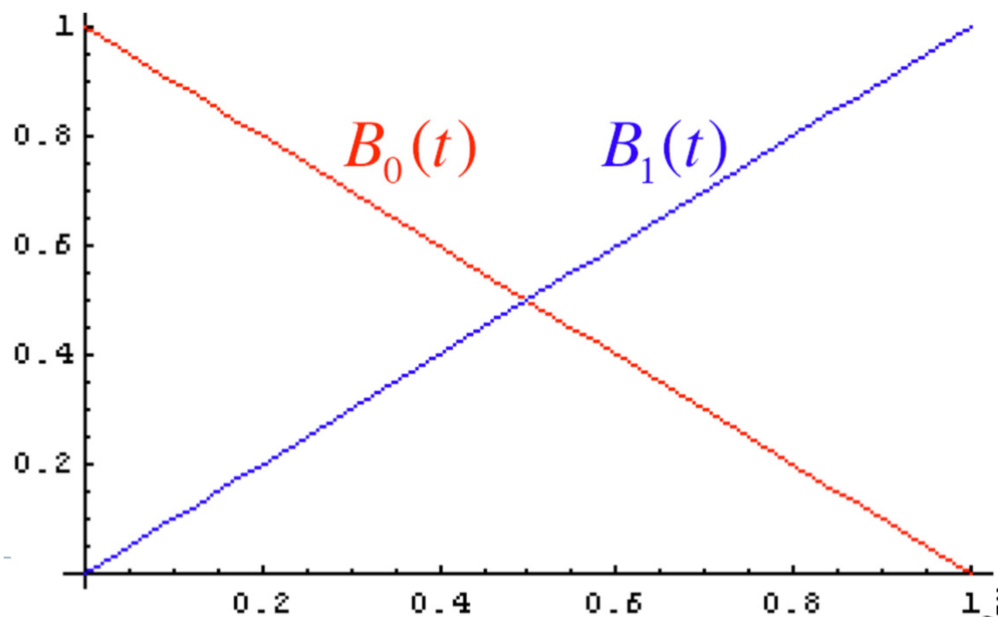
$$\mathbf{x}(t) = \begin{bmatrix} \mathbf{p}_0 & \mathbf{p}_1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} t \\ 1 \end{bmatrix}$$

Weighted Average

$$\mathbf{x}(t) = (1-t)\mathbf{p}_0 + \quad (t)\mathbf{p}_1$$

$$= B_0(t)\ \mathbf{p}_0 + B_1(t)\mathbf{p}_1,$$ where $B_0(t) = 1-t$ and $B_1(t) = t$

- Weights are a function of t
  - Sum is always 1, for any value of t
  - Also known as *blending functions*



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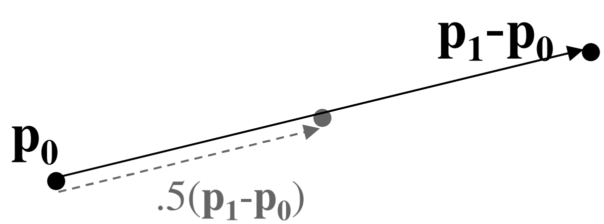
Linear Polynomial

$$\mathbf{x}(t) = \underbrace{(\mathbf{p}_1 - \mathbf{p}_0)}_{\substack{\text{vector} \\ \mathbf{a}}}\, t + \underbrace{\mathbf{p}_0}_{\substack{\text{point} \\ \mathbf{b}}}$$

- Curve is based at point $\mathbf{p_0}$
- Add the vector, scaled by t



Matrix Form

$$\mathbf{x}(t) = \begin{bmatrix} \mathbf{p}_0 & \mathbf{p}_1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} t \\ 1 \end{bmatrix} = \mathbf{GBT}$$

- Geometry matrix $\mathbf{G} = \begin{bmatrix} \mathbf{p}_0 & \mathbf{p}_1 \end{bmatrix}$
- Geometric basis $\mathbf{B} = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$
- Polynomial basis $T = \begin{bmatrix} t \\ 1 \end{bmatrix}$
- In components

$$\mathbf{x}(t) = \begin{bmatrix} p_{0x} & p_{1x} \\ p_{0y} & p_{1y} \\ p_{0z} & p_{1z} \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} t \\ 1 \end{bmatrix}$$

Matrix Form

$$\mathbf{x}(t) = \begin{bmatrix} \mathbf{p}_0 & \mathbf{p}_1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} t \\ 1 \end{bmatrix} = \mathbf{GBT}$$

- Geometry matrix $\mathbf{G} = \begin{bmatrix} \mathbf{p}_0 & \mathbf{p}_1 \end{bmatrix}$
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- In components

$$\mathbf{x}(t) = \begin{bmatrix} p_{0x} & p_{1x} \\ p_{0y} & p_{1y} \\ p_{0z} & p_{1z} \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} t \\ 1 \end{bmatrix}$$

Tangent

- For a straight line, the tangent is constant

$$\mathbf{x}'(t) = \mathbf{p}_1 - \mathbf{p}_0$$

- Weighted average $\mathbf{x}'(t) = (-1)\mathbf{p}_0 + (+1)\mathbf{p}_1$

- Polynomial $\mathbf{x}'(t) = 0t + (\mathbf{p}_1 - \mathbf{p}_0)$

- Matrix form $\mathbf{x}'(t) = \begin{bmatrix} \mathbf{p}_0 & \mathbf{p}_1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

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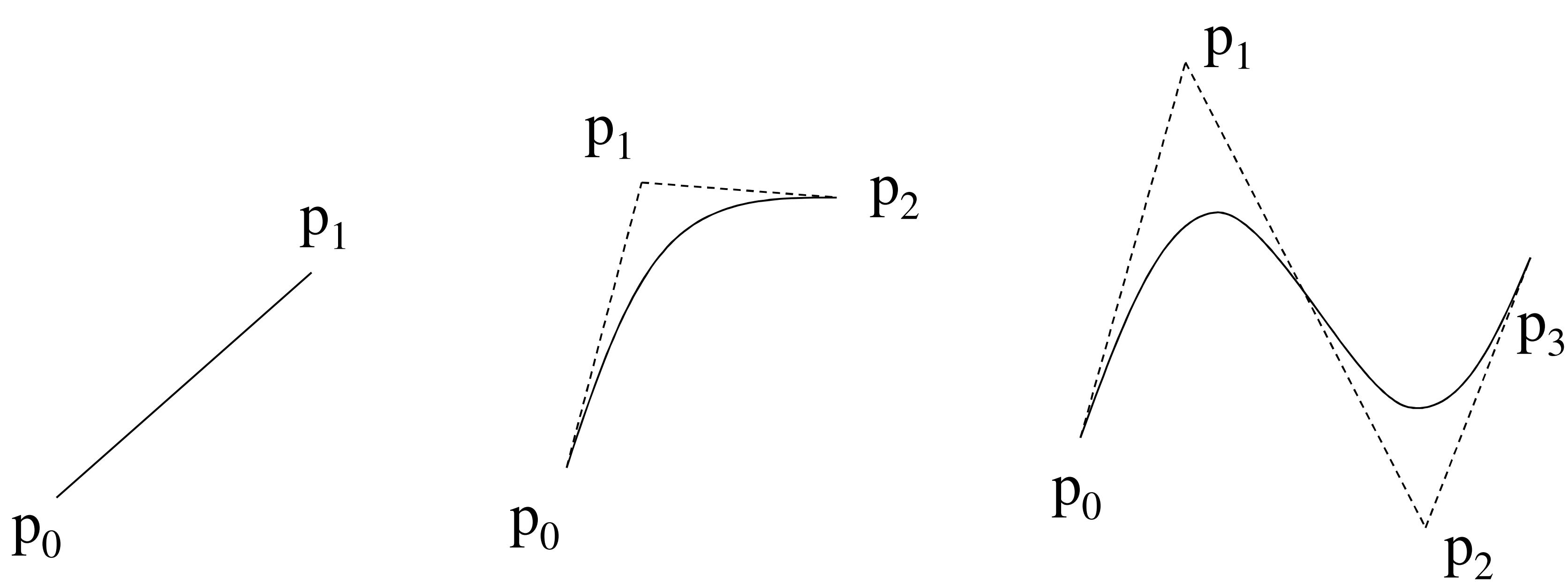
Bézier Curves

- Are a higher order extension of linear interpolation



Linear Quadratic Cubic

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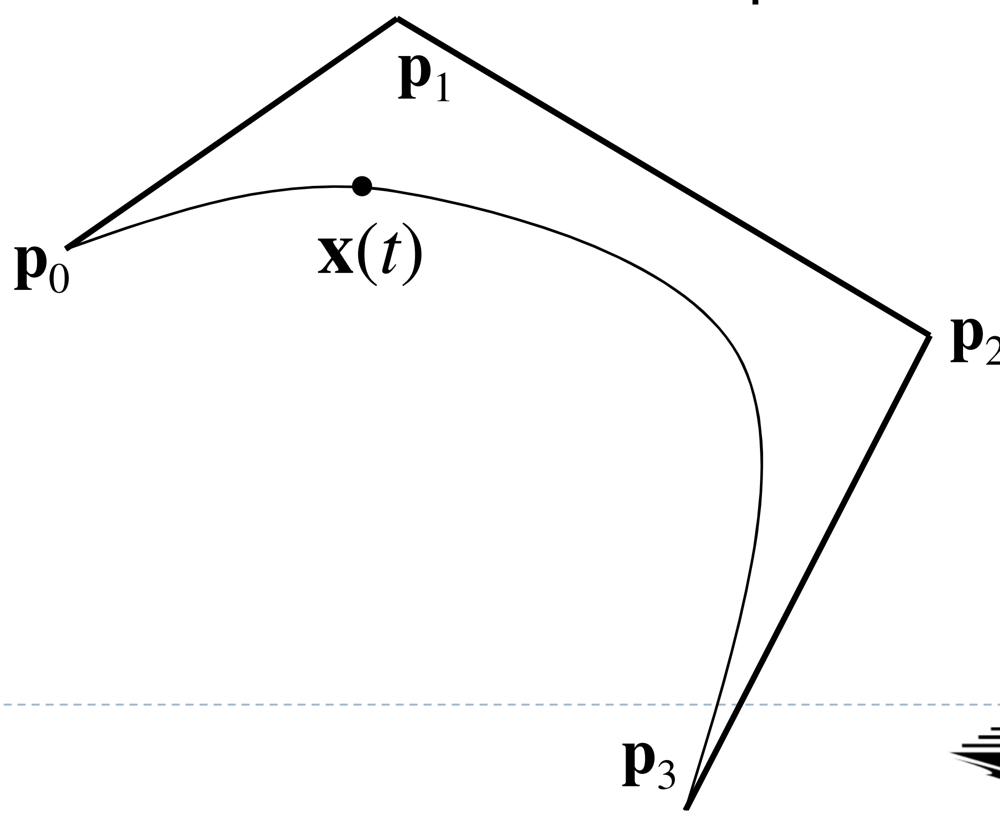
Bézier Curves

- Give intuitive control over curve with control points
  - Endpoints are interpolated, intermediate points are approximated
  - Convex Hull property
- Many demo applets online, for example:
  - Demo: http://www.cs.princeton.edu/~min/cs426/jar/bezier.html
  - http://www.theparticle.com/applets/nyu/BezierApplet/
  - http://www.sunsite.ubc.ca/LivingMathematics/V001N01/UBCExamples/Bezier/bezier.html

Cubic Bézier Curve

- Most commonly used case
- Defined by four control points:
  - Two interpolated endpoints (points are on the curve)
  - Two points control the tangents at the endpoints
- Points $\mathbf{x}$ on curve defined as function of parameter t



Algorithmic Construction

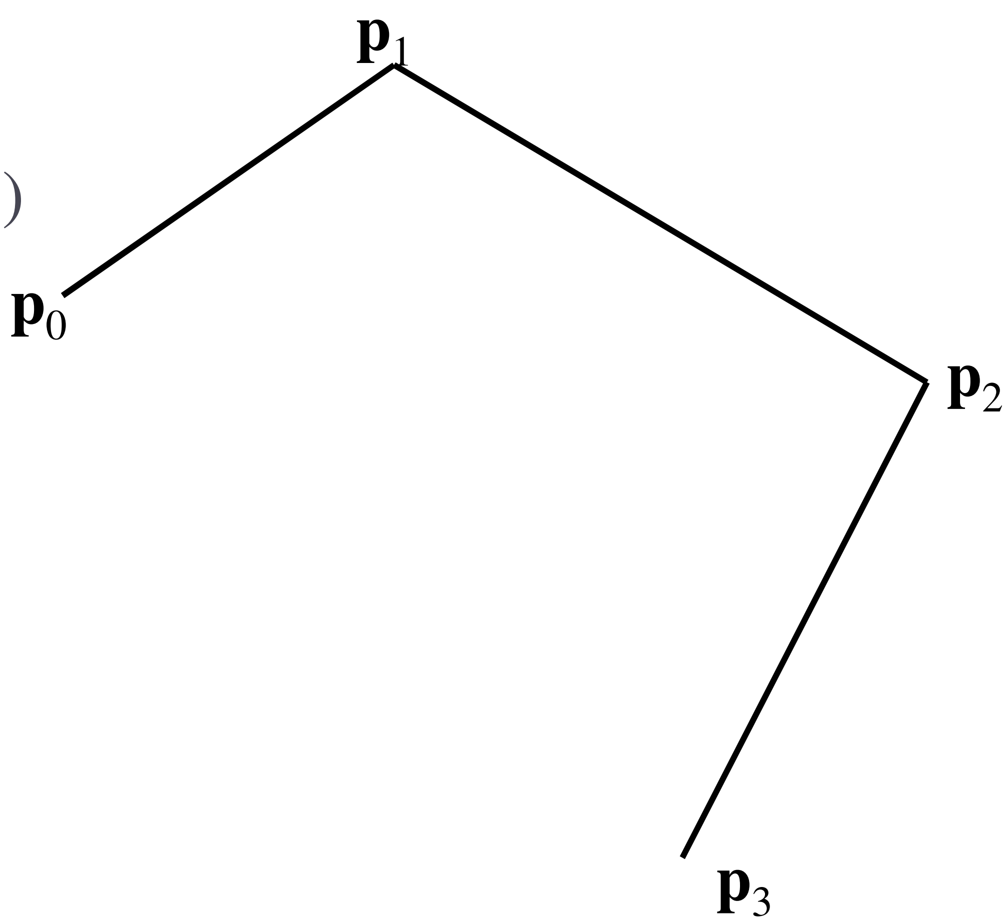
- Algorithmic construction
  - *De Casteljau* algorithm, developed at Citroen in 1959, named after its inventor Paul de Casteljau (pronounced "Cast-all-'Joe")
  - Developed independently from Bézier's work:
    Bézier created the formulation using blending functions, Casteljau devised the recursive interpolation algorithm

De Casteljau Algorithm

- A recursive series of linear interpolations
  - Works for any order Bezier function, not only cubic
- Not very efficient to evaluate
  - Other forms more commonly used
- But:
  - Gives intuition about the geometry
  - Useful for subdivision

De Casteljau Algorithm

- Given:
  - Four control points
  - A value of t (here $t \approx 0.25$)



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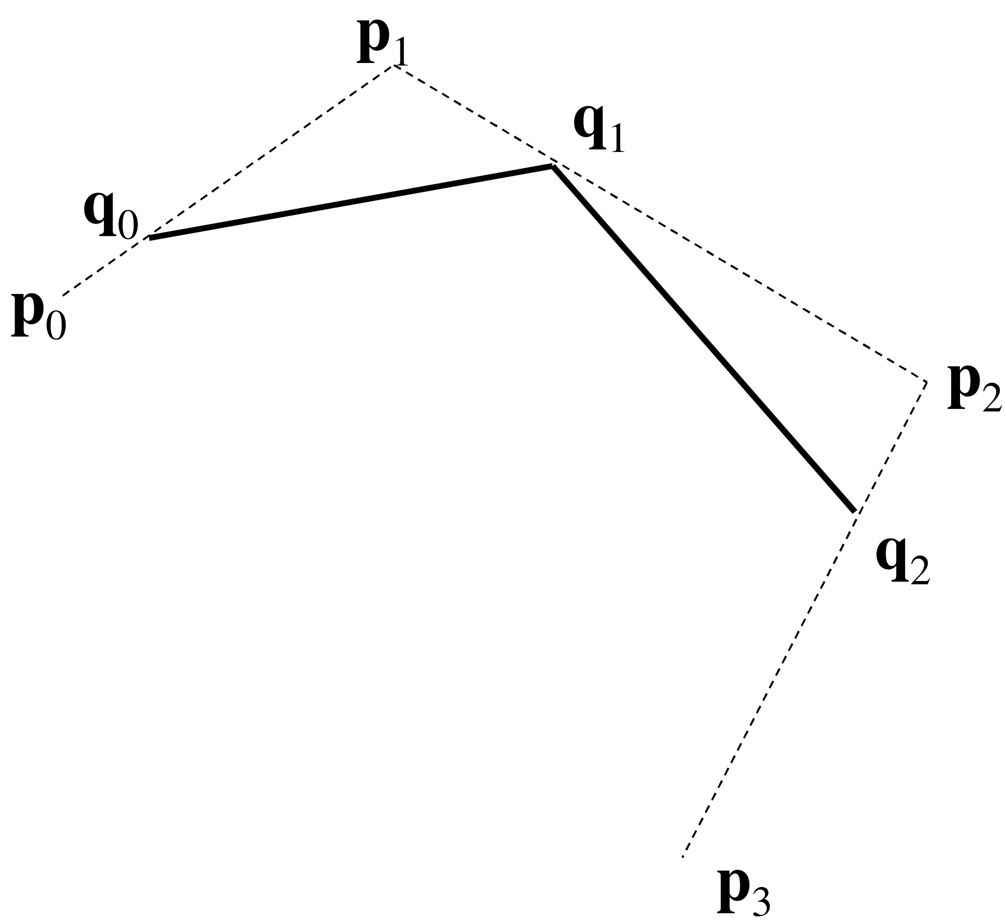
De Casteljau Algorithm

$$\mathbf{q}_0(t) = Lerp\left(t, \mathbf{p}_0, \mathbf{p}_1\right)$$

$$\mathbf{q}_1(t) = Lerp\left(t, \mathbf{p}_1, \mathbf{p}_2\right)$$

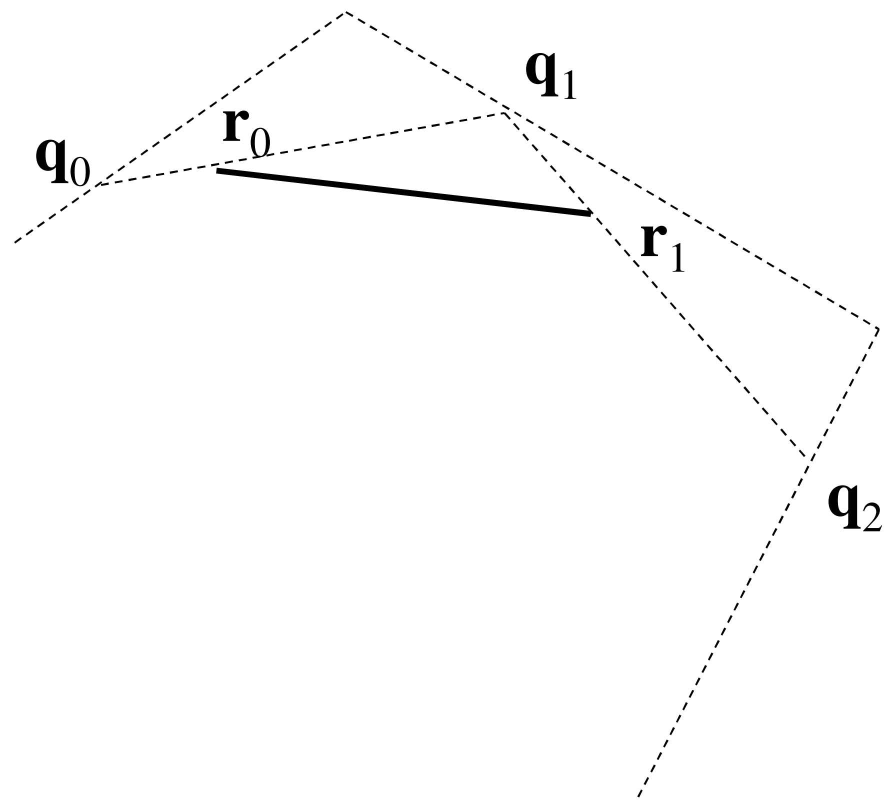
$$\mathbf{q}_2(t) = Lerp\left(t, \mathbf{p}_2, \mathbf{p}_3\right)$$



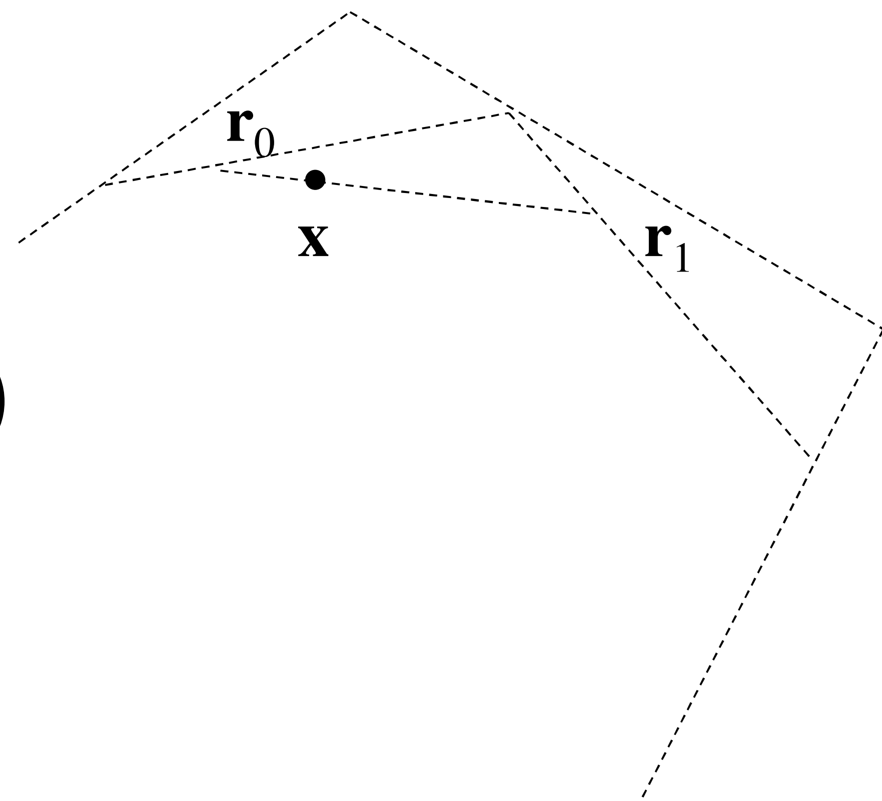
De Casteljau Algorithm

$$\mathbf{r}_0(t) = Lerp\big(t, \mathbf{q}_0(t), \mathbf{q}_1(t)\big)$$

$$\mathbf{r}_1(t) = Lerp\big(t, \mathbf{q}_1(t), \mathbf{q}_2(t)\big)$$

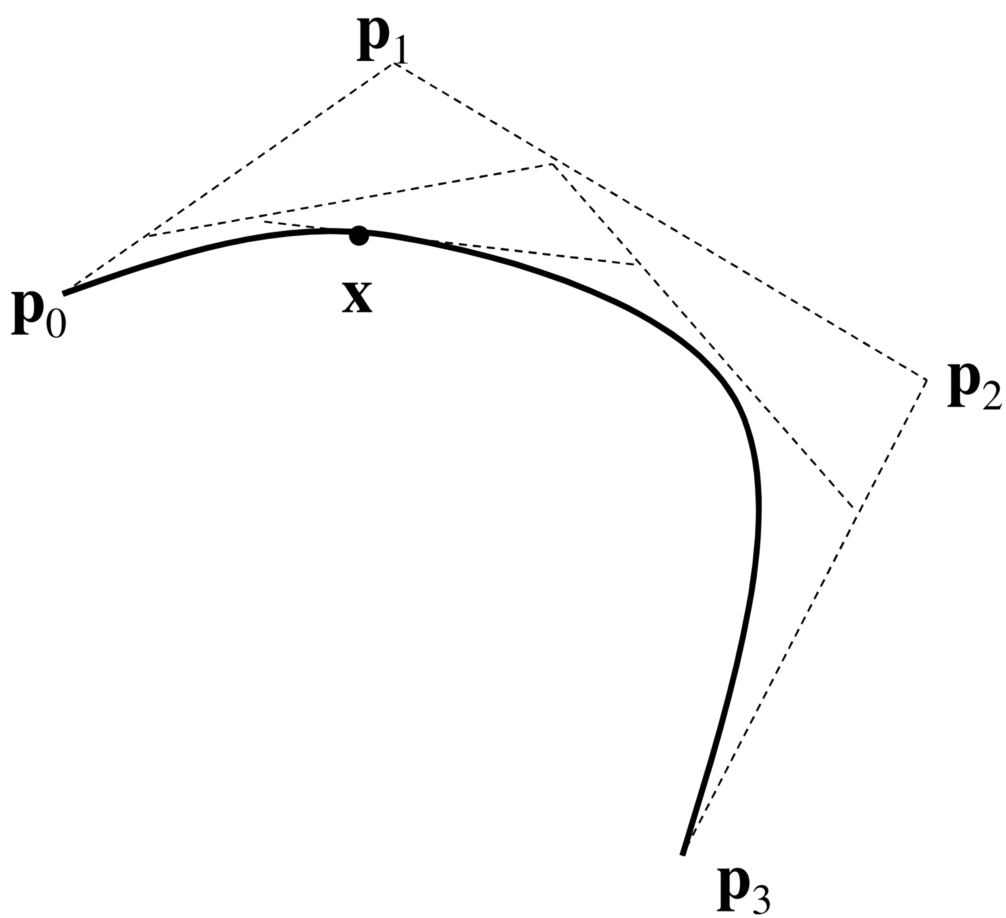


De Casteljau Algorithm



$$\mathbf{x}(t) = Lerp\big(t, \mathbf{r}_0(t), \mathbf{r}_1(t)\big)$$

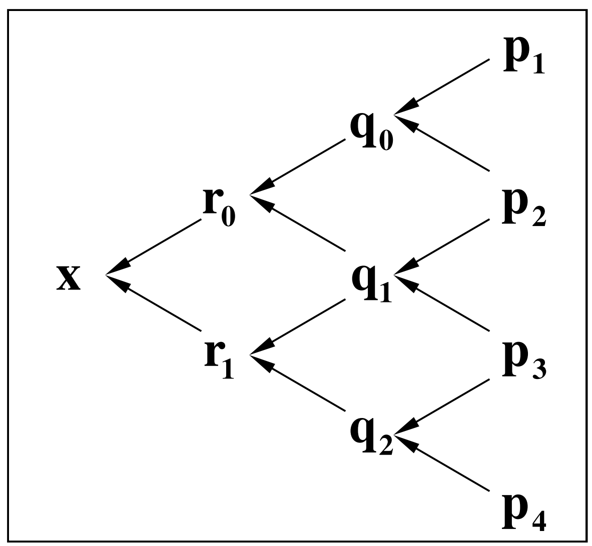
De Casteljau Algorithm



- Applets
  - Demo: http://www2.mat.dtu.dk/people/J.Gravesen/cagd/decast.html
  - http://www.caffeineowl.com/graphics/2d/vectorial/bezierintro.html

Recursive Linear Interpolation

$$\mathbf{x} = Lerp(t, \mathbf{r}_0, \mathbf{r}_1) \quad \begin{matrix} \mathbf{r}_0 = Lerp(t, \mathbf{q}_0, \mathbf{q}_1) \\ \mathbf{r}_1 = Lerp(t, \mathbf{q}_1, \mathbf{q}_2) \end{matrix} \quad \begin{matrix} \mathbf{q}_0 = Lerp(t, \mathbf{p}_0, \mathbf{p}_1) \\ \mathbf{q}_1 = Lerp(t, \mathbf{p}_1, \mathbf{p}_2) \\ \mathbf{q}_2 = Lerp(t, \mathbf{p}_2, \mathbf{p}_3) \end{matrix} \quad \begin{matrix} \mathbf{p}_0 \\ \mathbf{p}_1 \\ \mathbf{p}_2 \\ \mathbf{p}_3 \end{matrix}$$



Expand the LERPs

$$\mathbf{q}_0(t) = Lerp\left(t, \mathbf{p}_0, \mathbf{p}_1\right) = (1-t)\mathbf{p}_0 + t\mathbf{p}_1$$

$$\mathbf{q}_1(t) = Lerp\left(t, \mathbf{p}_1, \mathbf{p}_2\right) = (1-t)\mathbf{p}_1 + t\mathbf{p}_2$$

$$\mathbf{q}_2(t) = Lerp\left(t, \mathbf{p}_2, \mathbf{p}_3\right) = (1-t)\mathbf{p}_2 + t\mathbf{p}_3$$

$$\mathbf{r}_0(t) = Lerp\left(t, \mathbf{q}_0(t), \mathbf{q}_1(t)\right) = (1-t)\left((1-t)\mathbf{p}_0 + t\mathbf{p}_1\right) + t\left((1-t)\mathbf{p}_1 + t\mathbf{p}_2\right)$$

$$\mathbf{r}_1(t) = Lerp\left(t, \mathbf{q}_1(t), \mathbf{q}_2(t)\right) = (1-t)\left((1-t)\mathbf{p}_1 + t\mathbf{p}_2\right) + t\left((1-t)\mathbf{p}_2 + t\mathbf{p}_3\right)$$

$$\begin{aligned}\mathbf{x}(t) &= Lerp\left(t, \mathbf{r}_0(t), \mathbf{r}_1(t)\right) \\ &= (1-t)\Big((1-t)\left((1-t)\mathbf{p}_0 + t\mathbf{p}_1\right) + t\left((1-t)\mathbf{p}_1 + t\mathbf{p}_2\right)\Big) \\ &\qquad + t\Big((1-t)\left((1-t)\mathbf{p}_1 + t\mathbf{p}_2\right) + t\left((1-t)\mathbf{p}_2 + t\mathbf{p}_3\right)\Big)\end{aligned}$$

Weighted Average of Control Points

- Regroup for p:

$$\mathbf{x}(t) = (1-t)\Big((1-t)\big((1-t)\mathbf{p}_0 + t\mathbf{p}_1\big) + t\big((1-t)\mathbf{p}_1 + t\mathbf{p}_2\big)\Big)$$
$$+t\Big((1-t)\big((1-t)\mathbf{p}_1 + t\mathbf{p}_2\big) + t\big((1-t)\mathbf{p}_2 + t\mathbf{p}_3\big)\Big)$$

$$\mathbf{x}(t) = (1-t)^3\mathbf{p}_0 + 3(1-t)^2 t\mathbf{p}_1 + 3(1-t)t^2\mathbf{p}_2 + t^3\mathbf{p}_3$$

$$\mathbf{x}(t) = \overbrace{\left(-t^3 + 3t^2 - 3t + 1\right)}^{B_0(t)}\mathbf{p}_0 + \overbrace{\left(3t^3 - 6t^2 + 3t\right)}^{B_1(t)}\mathbf{p}_1$$
$$+\underbrace{\left(-3t^3 + 3t^2\right)}_{B_2(t)}\mathbf{p}_2 + \underbrace{\left(t^3\right)}_{B_3(t)}\mathbf{p}_3$$

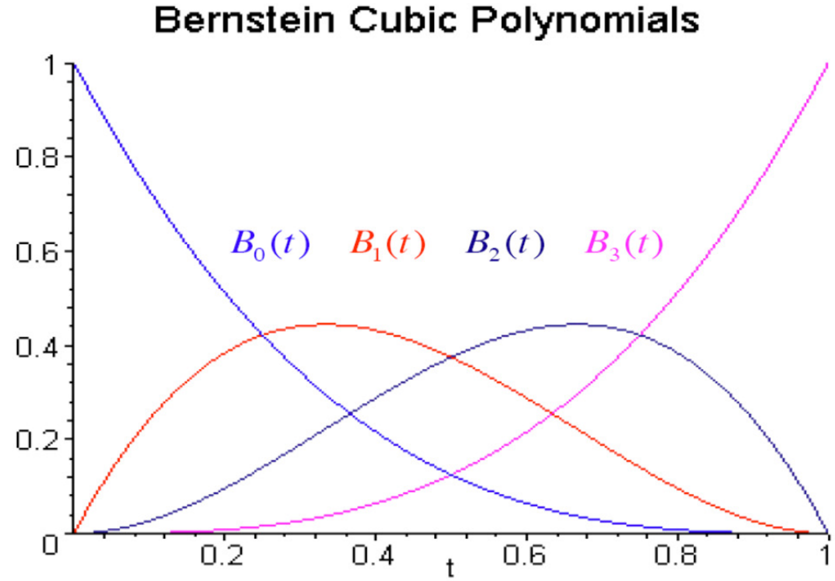
Cubic Bernstein Polynomials

$$\mathbf{x}(t) = B_0(t)\mathbf{p}_0 + B_1(t)\mathbf{p}_1 + B_2(t)\mathbf{p}_2 + B_3(t)\mathbf{p}_3$$

The cubic *Bernstein polynomials :*

$$B_0(t) = -t^3 + 3t^2 - 3t + 1$$
$$B_1(t) = 3t^3 - 6t^2 + 3t$$
$$B_2(t) = -3t^3 + 3t^2$$
$$B_3(t) = t^3$$

$$\sum B_i(t) = 1$$



- Weights $B_i(t)$ add up to 1 for any value of t

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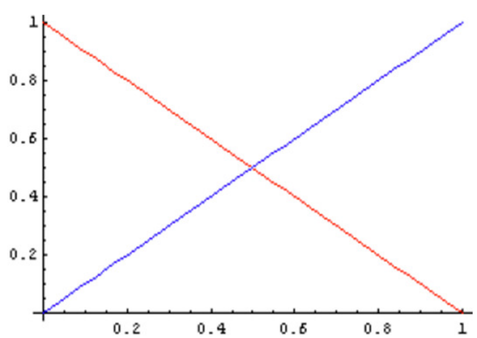
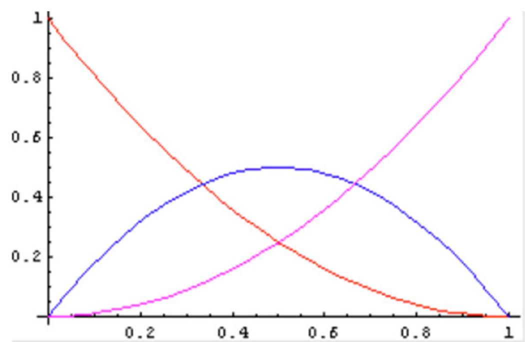
General Bernstein Polynomials

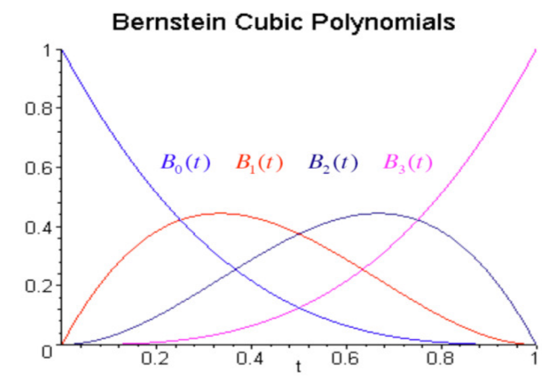
$$B_0^1(t) = -t + 1$$
$$B_1^1(t) = t$$

$$B_0^2(t) = t^2 - 2t + 1$$
$$B_1^2(t) = -2t^2 + 2t$$
$$B_2^2(t) = t^2$$

$$B_0^3(t) = -t^3 + 3t^2 - 3t + 1$$
$$B_1^3(t) = 3t^3 - 6t^2 + 3t$$
$$B_2^3(t) = -3t^3 + 3t^2$$
$$B_3^3(t) = t^3$$







$$B_i^n(t) = \binom{n}{i} (1-t)^{n-i} (t)^i \qquad \binom{n}{i} = \frac{n!}{i!(n-i)!}$$

$$\sum B_i^n(t) = 1$$

n! = factorial of n
(n+1)! = n! x (n+1)

General Bézier Curves

- nth-order Bernstein polynomials form nth-order Bézier curves

$$B_i^n(t) = \binom{n}{i} (1-t)^{n-i} (t)^i$$

$$\mathbf{x}(t) = \sum_{i=0}^{n} B_i^n(t) \mathbf{p}_i$$