CSE 167:
Introduction to Computer Graphics
Lecture #3: Linear Algebra

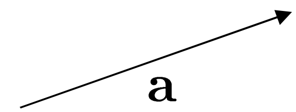
Jürgen P. Schulze, Ph.D.
University of California, San Diego
Fall Quarter 2018

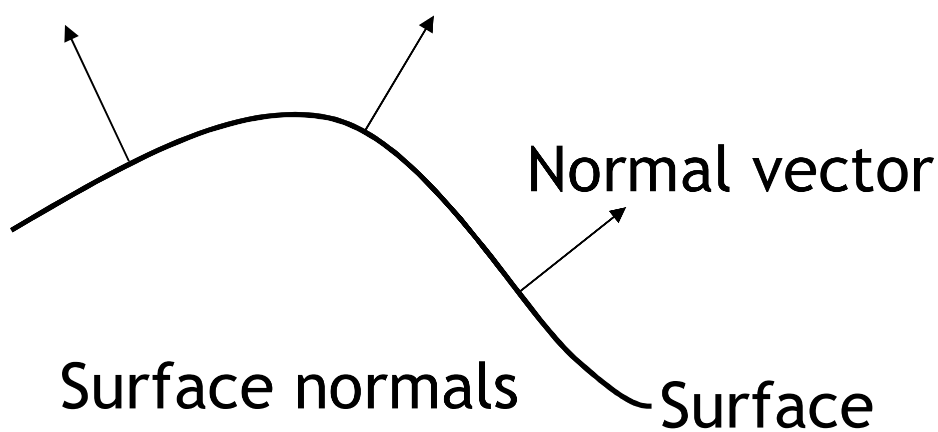
Overview

- Vectors and matrices
- Affine transformations
- Homogeneous coordinates

Vectors

- Give direction and length in 3D
- Vectors can describe
  - Difference between two 3D points
  - Speed of an object
  - Surface normals (directions perpendicular to surfaces)





Vector arithmetic using coordinates

$$\mathbf{a} = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix}$$

$$\mathbf{a} + \mathbf{b} = \begin{bmatrix} a_x + b_x \\ a_y + b_y \\ a_z + b_z \end{bmatrix} \qquad \mathbf{a} - \mathbf{b} = \begin{bmatrix} a_x - b_x \\ a_y - b_y \\ a_z - b_z \end{bmatrix}$$

$$-\mathbf{a} = \begin{bmatrix} -a_x \\ -a_y \\ -a_z \end{bmatrix} \qquad s\mathbf{a} = \begin{bmatrix} sa_x \\ sa_y \\ sa_z \end{bmatrix}$$

where s is a scalar

Vector Magnitude

- The magnitude (length) of a vector is:

$$|\mathbf{v}|^2 = v_x^2 + v_y^2 + v_z^2$$

$$|\mathbf{v}| = \sqrt{v_x^2 + v_y^2 + v_z^2}$$

- A vector with length of 1.0 is called *unit vector*
- We can also *normalize* a vector to make it a unit vector

$$\frac{\mathbf{v}}{|\mathbf{v}|}$$

- Unit vectors are often used as **surface normals**

Dot Product

$$\mathbf{a} \cdot \mathbf{b} = \sum a_i b_i$$

$$\mathbf{a} \cdot \mathbf{b} = a_x b_x + a_y b_y + a_z b_z$$

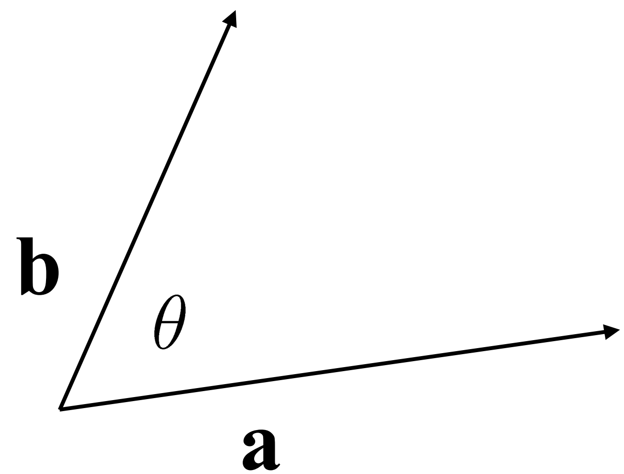
$$\mathbf{a} \cdot \mathbf{b} = |a||b| \cos\theta$$

Angle Between Two Vectors

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$$

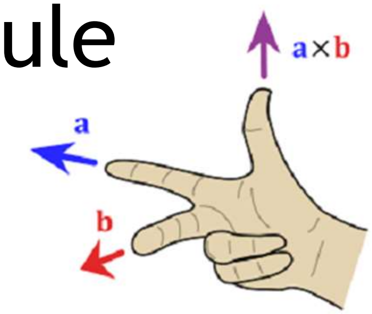
$$\cos\theta = \left(\frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|} \right)$$

$$\theta = \cos^{-1}\left(\frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|} \right)$$



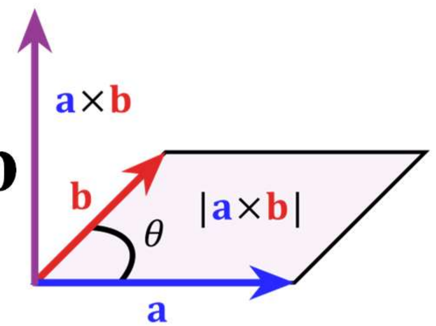
Cross Product

$\mathbf{a} \times \mathbf{b}$ is a vector *perpendicular* to both $\mathbf{a}$ and $\mathbf{b}$, in the direction defined by the right hand rule



$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin\theta$$

$|\mathbf{a} \times \mathbf{b}| =$ area of parallelogram $\mathbf{ab}$



$|\mathbf{a} \times \mathbf{b}| = 0$ if $\mathbf{a}$ and $\mathbf{b}$ are parallel (or one or both degenerate)

Cross Product

$$a \times b = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} \times \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix} = \begin{bmatrix} a_y b_z - a_z b_y \\ a_z b_x - a_x b_z \\ a_x b_y - a_y b_x \end{bmatrix}$$

Cross Product Calculation

$$\begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} \times \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix} = \begin{bmatrix} \boxed{a_y b_z - a_z b_y} \\ a_z b_x - a_x b_z \\ a_x b_y - a_y b_x \end{bmatrix}$$

$$\begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} \times \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix} = \begin{bmatrix} a_y b_z - a_z b_y \\ \boxed{a_z b_x - a_x b_z} \\ a_x b_y - a_y b_x \end{bmatrix}$$

$$\begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} \times \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix} = \begin{bmatrix} a_y b_z - a_z b_y \\ a_z b_x - a_x b_z \\ \boxed{a_x b_y - a_y b_x} \end{bmatrix}$$

Matrices

- Rectangular array of numbers

$$\mathbf{M} = \begin{bmatrix} m_{1,1} & m_{1,2} & \ldots & m_{1,n} \\ m_{2,1} & m_{2,2} & \ldots & m_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ m_{m,1} & m_{2,2} & \ldots & m_{m,n} \end{bmatrix} \in \mathbf{R}^{m \times n}$$

- Square matrix if **m = n**
- In graphics almost always: **m = n = 3; m = n = 4**

Matrix Addition

$$\mathbf{A}+\mathbf{B}=\begin{bmatrix} a_{1,1}+b_{1,1} & a_{1,2}+b_{1,2} & \dots & a_{1,n}+b_{1,n} \\ a_{2,1}+b_{2,1} & a_{2,2}+b_{2,2} & \dots & a_{2,n}+b_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1}+b_{m,1} & a_{2,2}+b_{2,2} & \dots & a_{m,n}+b_{m,n} \end{bmatrix}$$

$$\mathbf{A},\mathbf{B}\in\mathbf{R}^{m\times n}$$

Multiplication With Scalar

$$s\mathbf{M} = \mathbf{M}s = \begin{bmatrix} sm_{1,1} & sm_{1,2} & \dots & sm_{1,n} \\ sm_{2,1} & sm_{2,2} & \dots & sm_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ sm_{m,1} & sm_{2,2} & \dots & sm_{m,n} \end{bmatrix}$$

Matrix Multiplication

$$\mathbf{AB} = \mathbf{C}, \quad \mathbf{A} \in \mathbf{R}^{p,q}, \mathbf{B} \in \mathbf{R}^{q,r}, \mathbf{C} \in \mathbf{R}^{p,r}$$

$$(\mathbf{AB})_{i,j} = \mathbf{C}_{i,j} = \sum_{k=1}^{q} a_{i,k} b_{k,j}, \quad i \in 1..p, j \in 1..r$$

Matrix-Vector Multiplication

$$\begin{bmatrix} \mathrm{a} & b \\ \mathrm{c} & d \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$$

$$\begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} ax + by + c \\ dx + ey + f \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} ax + by + cz \\ dx + ey + fz \\ gx + hy + iz \end{bmatrix}$$

$$\begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} ax + by + cz + d \\ ex + fy + gz + h \\ ix + jy + kz + l \\ 1 \end{bmatrix}$$

Identity Matrix

$$I_1 = [1],\ I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},\ \cdots,\ I_n = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

$$\mathbf{MI} = \mathbf{IM} = \mathbf{M}, \quad \text{for any } \mathbf{M} \in \mathbf{R}^{n \times n}$$

Matrix Inverse

If a square matrix **M** is non-singular, there exists a unique inverse $\mathbf{M}^{-1}$ such that

$$\mathbf{M}\mathbf{M}^{-1} = \mathbf{M}^{-1}\mathbf{M} = \mathbf{I}$$

$$(\mathbf{MPQ})^{-1} = \mathbf{Q}^{-1}\mathbf{P}^{-1}\mathbf{M}^{-1}$$

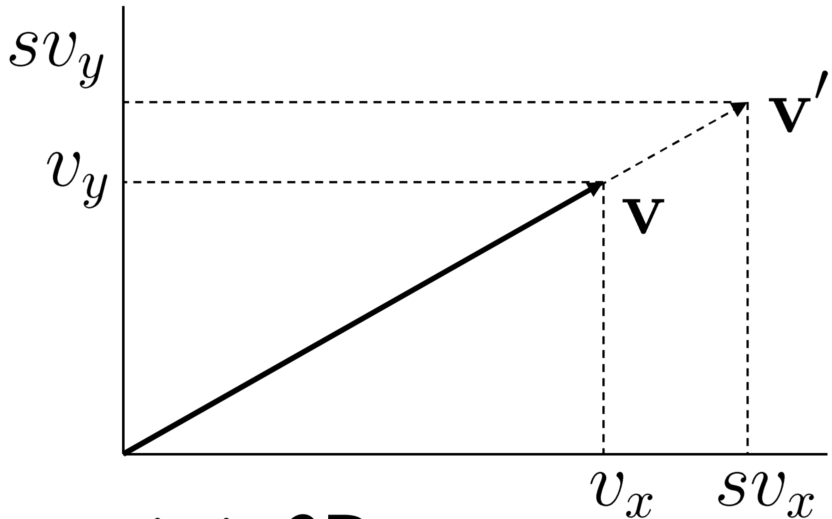
Overview

- Vectors and matrices
- Affine transformations
- Homogeneous coordinates

Affine Transformations

- Most important for graphics:
  - rotation, translation, scaling
- Wolfram MathWorld:
  - An **affine transformation** is any **transformation** that preserves collinearity (i.e., all points lying on a line initially still lie on a line after **transformation**) and ratios of distances (e.g., the midpoint of a line segment remains the midpoint after **transformation**).
- Implemented using matrix multiplications

Uniform Scale

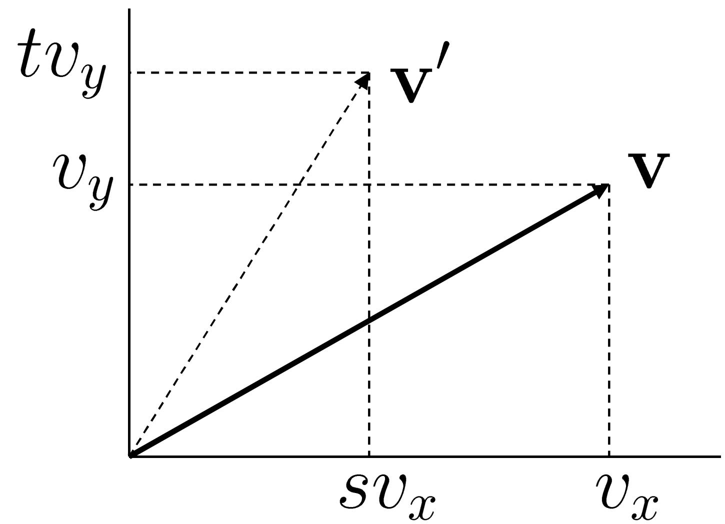


- Uniform scale matrix in 2D

$$\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} \mathbf{v} = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} \begin{bmatrix} v_x \\ v_y \end{bmatrix} = \begin{bmatrix} v'_x \\ v'_y \end{bmatrix} = \mathbf{v}'$$

- Analogous in 3D: $\begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix}$

Non-Uniform Scale



- Nonuniform scaling matrix in 2D

$$\begin{bmatrix} s & 0 \\ 0 & t \end{bmatrix} \mathbf{v} = \begin{bmatrix} s & 0 \\ 0 & t \end{bmatrix} \begin{bmatrix} v_x \\ v_y \end{bmatrix} = \begin{bmatrix} v'_x \\ v'_y \end{bmatrix} = \mathbf{v}'$$

Non-Uniform Scale in 3D

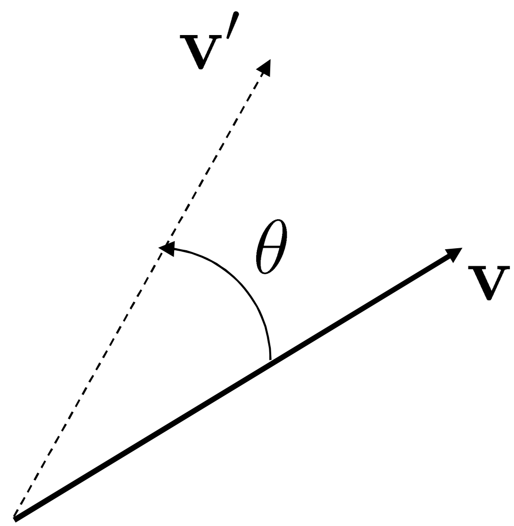
- Scale in 2D: $\begin{bmatrix} s & 0 \\ 0 & t \end{bmatrix}$

- Analogous in 3D: $\begin{bmatrix} s & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & u \end{bmatrix}$

Rotation in 2D

- Convention: positive angle rotates counterclockwise
- Rotation matrix

$$\mathbf{R}(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$



$$\mathbf{v}' = \mathbf{R}(\theta)\mathbf{v}$$

Rotation in 3D

Rotation around coordinate axes

$$\mathbf{R}_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

$$\mathbf{R}_y(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$\mathbf{R}_z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Rotation in 3D

- Concatenation of rotations around x, y, z axes

$$\mathbf{R}_{x,y,z}(\theta_x, \theta_y, \theta_z) = \mathbf{R}_x(\theta_x)\mathbf{R}_y(\theta_y)\mathbf{R}_z(\theta_z)$$

- $\theta_x, \theta_y, \theta_z$ are called Euler angles
- Result depends on matrix order!

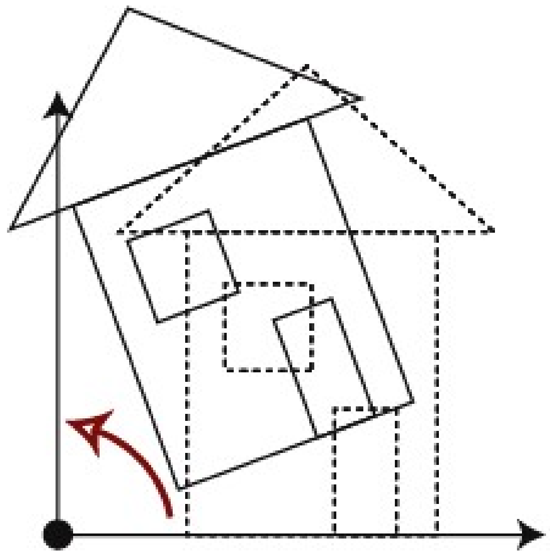
$$\mathbf{R}_x(\theta_x)\mathbf{R}_y(\theta_y)\mathbf{R}_z(\theta_z) \neq \mathbf{R}_z(\theta_z)\mathbf{R}_y(\theta_y)\mathbf{R}_x(\theta_x)$$

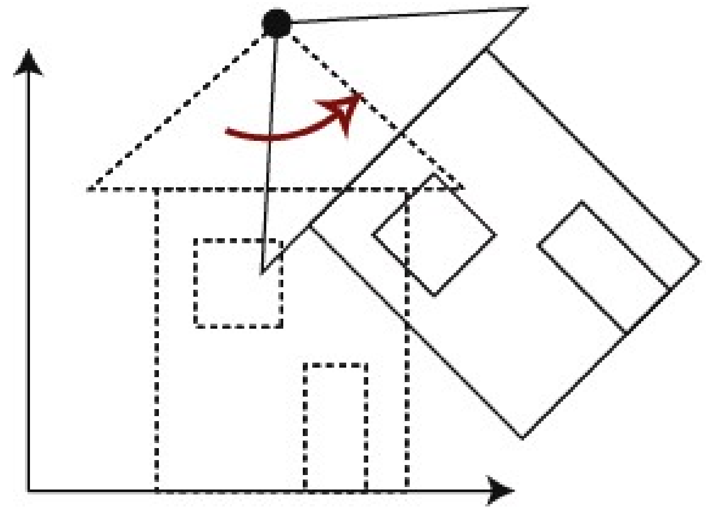
Rotation about an Arbitrary Axis

- Complicated!
- Rotate point [x,y,z] about axis [u,v,w] by angle θ:

$$\begin{bmatrix} \dfrac{u(ux+vy+wz)(1-\cos\theta)+(u^2+v^2+w^2)x\cos\theta+\sqrt{u^2+v^2+w^2}(-wy+vz)\sin\theta}{u^2+v^2+w^2} \\ \dfrac{v(ux+vy+wz)(1-\cos\theta)+(u^2+v^2+w^2)y\cos\theta+\sqrt{u^2+v^2+w^2}(wx-uz)\sin\theta}{u^2+v^2+w^2} \\ \dfrac{w(ux+vy+wz)(1-\cos\theta)+(u^2+v^2+w^2)z\cos\theta+\sqrt{u^2+v^2+w^2}(-vx+uy)\sin\theta}{u^2+v^2+w^2} \end{bmatrix}$$

How to rotate around a Pivot Point?

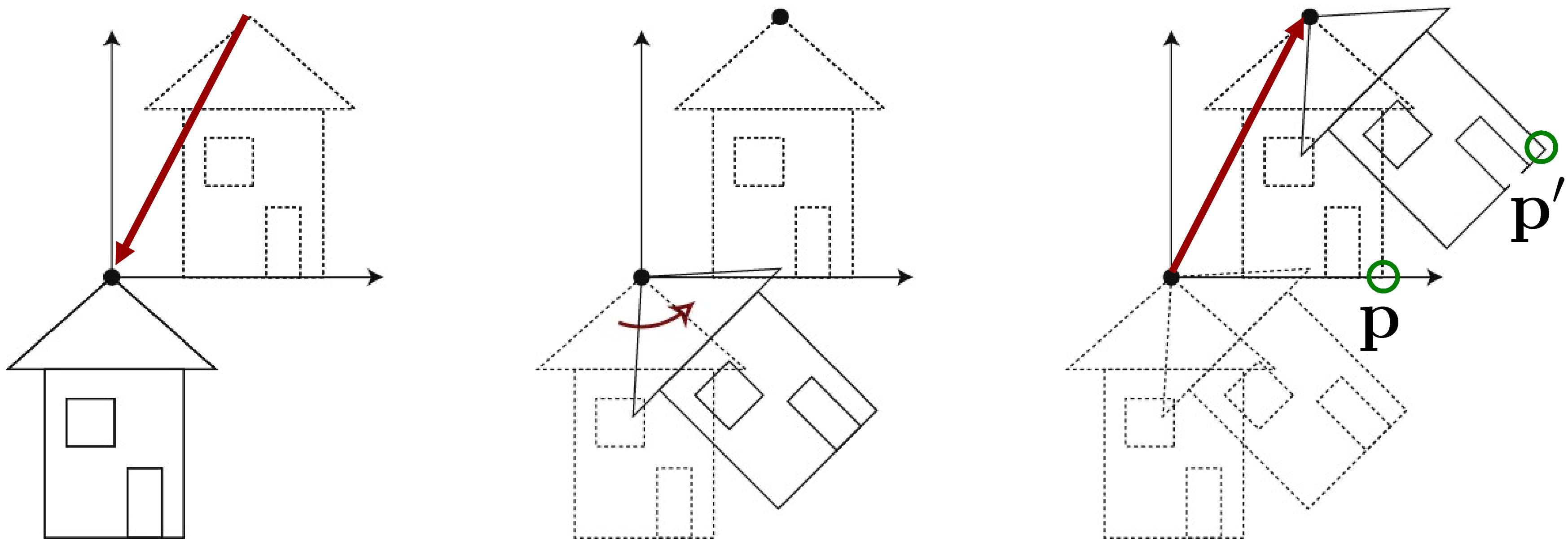




Rotation around origin:
p' = R p

Rotation around pivot point:
p' = ?

Rotating point p around a pivot point



1. Translation T 2. Rotation R 3. Translation T^{-1}

$$p' = T^{-1}\,R\,T\,p$$

Concatenating transformations

- Given a sequence of transformations $\mathbf{M}_3\mathbf{M}_2\mathbf{M}_1$

$$\mathbf{p}' = \mathbf{M}_3\mathbf{M}_2\mathbf{M}_1\mathbf{p}$$

$$\mathbf{M}_{total} = \mathbf{M}_3\mathbf{M}_2\mathbf{M}_1$$

$$\mathbf{p}' = \mathbf{M}_{total}\mathbf{p}$$

- Note: associativity applies

$$\mathbf{M}_{total} = (\mathbf{M}_3\mathbf{M}_2)\mathbf{M}_1 = \mathbf{M}_3(\mathbf{M}_2\mathbf{M}_1)$$

Overview

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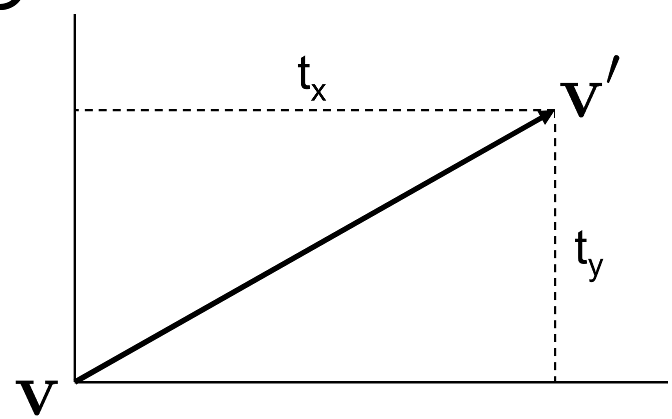
Translation

- Translation in 2D



- Translation matrix T=?

$$v' = \begin{bmatrix} v_x \\ v_y \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \end{bmatrix} = Tv = T\begin{bmatrix} v_x \\ v_y \end{bmatrix} = \begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix}\begin{bmatrix} v_x \\ v_y \end{bmatrix}$$

Translation

- Translation in 2D: 3x3 matrix

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

- Analogous in 3D: 4x4 matrix

$$\begin{bmatrix} x' \\ y' \\ z' \\ w \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

Homogeneous Coordinates

- Basic: a trick to unify/simplify computations.

- Deeper: projective geometry
  - Interesting mathematical properties
  - Good to know, but less immediately practical
  - We will use some aspect of this when we do perspective projection

Homogeneous Coordinates

- Add an extra component. 1 for a point, 0 for a vector:

$$\mathbf{p} = \begin{bmatrix} p_x \\ p_y \\ p_z \\ 1 \end{bmatrix} \qquad \vec{\mathbf{v}} = \begin{bmatrix} v_x \\ v_y \\ v_z \\ 0 \end{bmatrix}$$

- Combine **M** and **d** into single 4x4 matrix:

$$\begin{bmatrix} m_{xx} & m_{xy} & m_{xz} & d_x \\ m_{yx} & m_{yy} & m_{yz} & d_y \\ m_{zx} & m_{zy} & m_{zz} & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- And see what happens when we multiply…

Homogeneous Point Transform

- Transform a point:

$$\begin{bmatrix} p'_x \\ p'_y \\ p'_z \\ 1 \end{bmatrix} = \begin{bmatrix} m_{xx} & m_{xy} & m_{xz} & d_x \\ m_{yx} & m_{yy} & m_{yz} & d_y \\ m_{zx} & m_{zy} & m_{zz} & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ p_z \\ 1 \end{bmatrix} = \begin{bmatrix} m_{xx}p_x + m_{xy}p_y + m_{xz}p_z + d_x \\ m_{yx}p_x + m_{yy}p_y + m_{yz}p_z + d_y \\ m_{zx}p_x + m_{zy}p_y + m_{zz}p_z + d_z \\ 0 + 0 + 0 + 1 \end{bmatrix}$$

$$\mathbf{M} \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} + \vec{\mathbf{d}}$$

- Top three rows are the affine transform!
- Bottom row stays 1

Homogeneous Vector Transform

- Transform a vector:

$$\begin{bmatrix} v'_x \\ v'_y \\ v'_z \\ 0 \end{bmatrix} = \begin{bmatrix} m_{xx} & m_{xy} & m_{xz} & d_x \\ m_{yx} & m_{yy} & m_{yz} & d_y \\ m_{zx} & m_{zy} & m_{zz} & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_x \\ v_y \\ v_z \\ 0 \end{bmatrix} = \begin{bmatrix} \boxed{\begin{matrix} m_{xx}v_x + m_{xy}v_y + m_{xz}v_z \\ m_{yx}v_x + m_{yy}v_y + m_{yz}v_z \\ m_{zx}v_x + m_{zy}v_y + m_{zz}v_z \end{matrix}} \begin{matrix} +0 \\ +0 \\ +0 \end{matrix} \\ 0+0+0+0 \end{bmatrix}$$

$$\mathbf{M} \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}$$

- Top three rows are the linear transform
  - Displacement **d** is properly ignored
- Bottom row stays 0

Homogeneous Arithmetic

- Legal operations always end in 0 or 1!

$$\text{vector+vector:} \quad \begin{bmatrix} \vdots \\ 0 \end{bmatrix} + \begin{bmatrix} \vdots \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} \vdots \\ 0 \end{bmatrix}$$

$$\text{vector-vector:} \quad \begin{bmatrix} \vdots \\ 0 \end{bmatrix} - \begin{bmatrix} \vdots \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} \vdots \\ 0 \end{bmatrix}$$

$$\text{scalar*vector:} \quad s\begin{bmatrix} \vdots \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} \vdots \\ 0 \end{bmatrix}$$

$$\text{point+vector:} \quad \begin{bmatrix} \vdots \\ 1 \end{bmatrix} + \begin{bmatrix} \vdots \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} \vdots \\ 1 \end{bmatrix}$$

$$\text{point-point:} \quad \begin{bmatrix} \vdots \\ 1 \end{bmatrix} - \begin{bmatrix} \vdots \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} \vdots \\ 0 \end{bmatrix}$$

$$\text{point+point:} \quad \begin{bmatrix} \vdots \\ 1 \end{bmatrix} + \begin{bmatrix} \vdots \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} \vdots \\ 2 \end{bmatrix}$$

$$\text{scalar*point:} \quad s\begin{bmatrix} \vdots \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} \vdots \\ s \end{bmatrix}$$

$$\left\{ \begin{matrix} \text{weighted average} \\ \text{affine combination} \end{matrix} \right\} \text{of points:} \quad \frac{1}{3}\begin{bmatrix} \vdots \\ 1 \end{bmatrix} + \frac{2}{3}\begin{bmatrix} \vdots \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} \vdots \\ 1 \end{bmatrix}$$

Homogeneous Transforms

- Rotation, Scale, and Translation of points and vectors unified in a single matrix transformation:

$$\mathbf{p}' = \mathbf{M}\ \mathbf{p}$$

- Matrix has the form:
  - Last row always 0,0,0,1

$$\begin{bmatrix} m_{xx} & m_{xy} & m_{xz} & d_x \\ m_{yx} & m_{yy} & m_{yz} & d_y \\ m_{zx} & m_{zy} & m_{zz} & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Transforms can be composed by matrix multiplication
  - Same caveat: order of operations is important
  - Same note: transforms operate right-to-left

4x4 Scale Matrix

- Generic form:

$$\begin{bmatrix} s & 0 & 0 & 0 \\ 0 & t & 0 & 0 \\ 0 & 0 & u & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Inverse:

$$\begin{bmatrix} \frac{1}{s} & 0 & 0 & 0 \\ 0 & \frac{1}{t} & 0 & 0 \\ 0 & 0 & \frac{1}{u} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

4x4 Rotation Matrix

- Generic form:

$$\begin{bmatrix} r_1 & r_2 & r_3 & 0 \\ r_4 & r_5 & r_6 & 0 \\ r_7 & r_8 & r_9 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Inverse:

$$\begin{bmatrix} r_1 & r_4 & r_7 & 0 \\ r_2 & r_5 & r_8 & 0 \\ r_3 & r_6 & r_9 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

4x4 Translation Matrix

- Generic form: $$\begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Inverse: $$\begin{bmatrix} 1 & 0 & 0 & -t_x \\ 0 & 1 & 0 & -t_y \\ 0 & 0 & 1 & -t_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$