CSE 167: Introduction to Computer Graphics Lecture 12: Bézier Curves

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Announcements

- Homework assignment 5 due tomorrow, Nov 8 at 1:30pm
- Late submissions for assignment 4 will be accepted
- CSE 169: Computer Animation
  - Most recent course web site from 2009 is:
    - http://graphics.ucsd.edu/courses/cse169_w09
  - PixelActive's CityScape:
    - http://www.youtube.com/watch?v=yrqm9qK_MIo

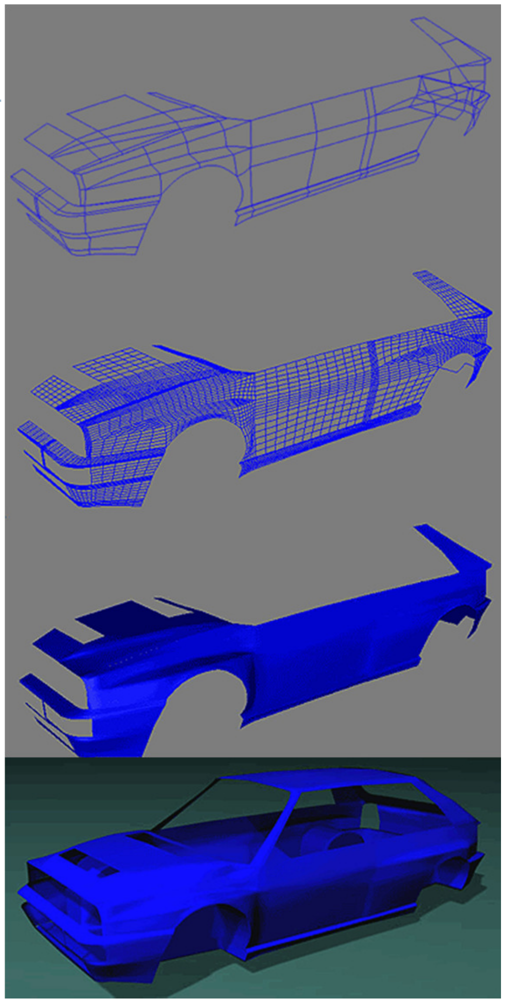
Lecture Overview

- Polynomial Curves
  - Introduction
  - Polynomial functions
- Bézier Curves
  - Introduction
  - Drawing Bézier curves
  - Piecewise Bézier curves

Modeling

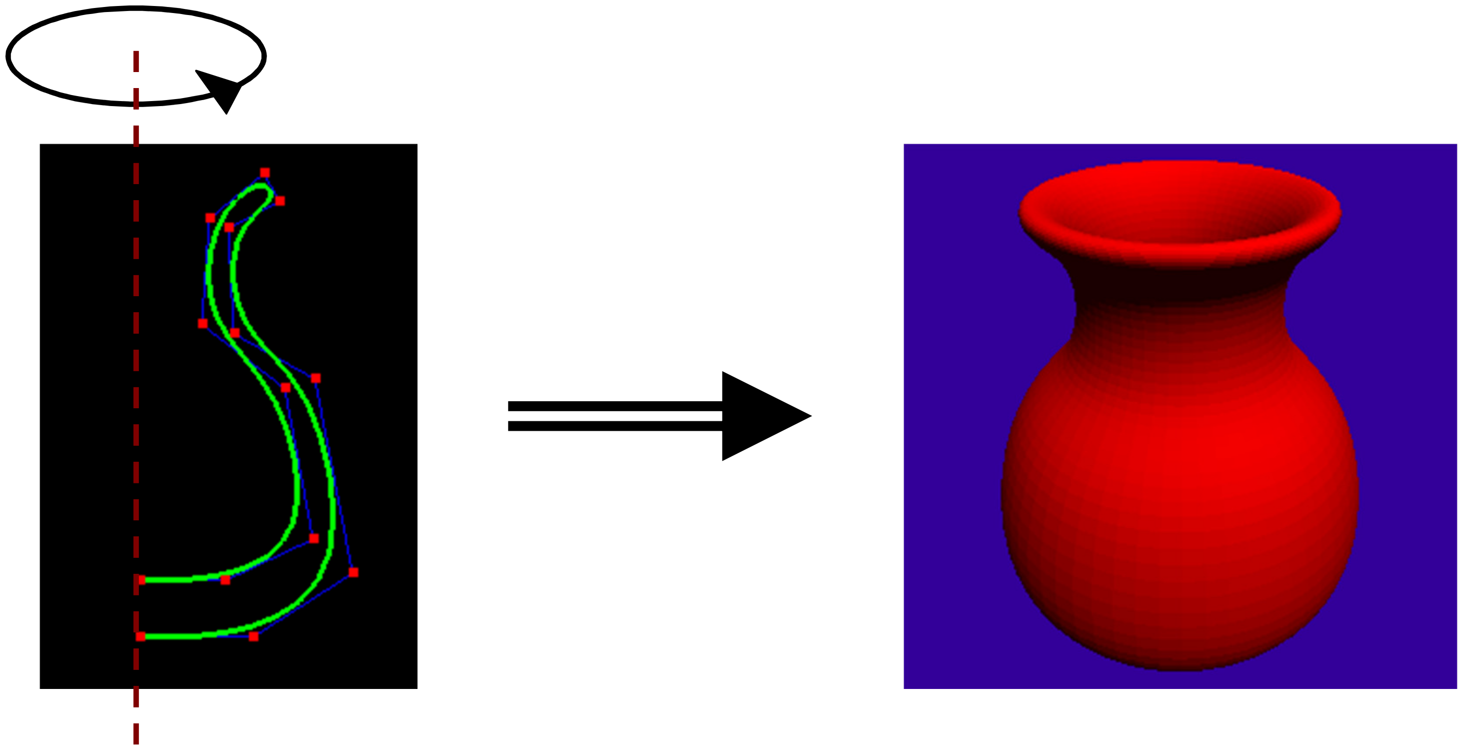
- Creating 3D objects
- How to construct complex surfaces?
- Goal
  - Specify objects with control points
  - Objects should be visually pleasing (smooth)
- Start with curves, then generalize to surfaces

- Next: What can curves be used for?



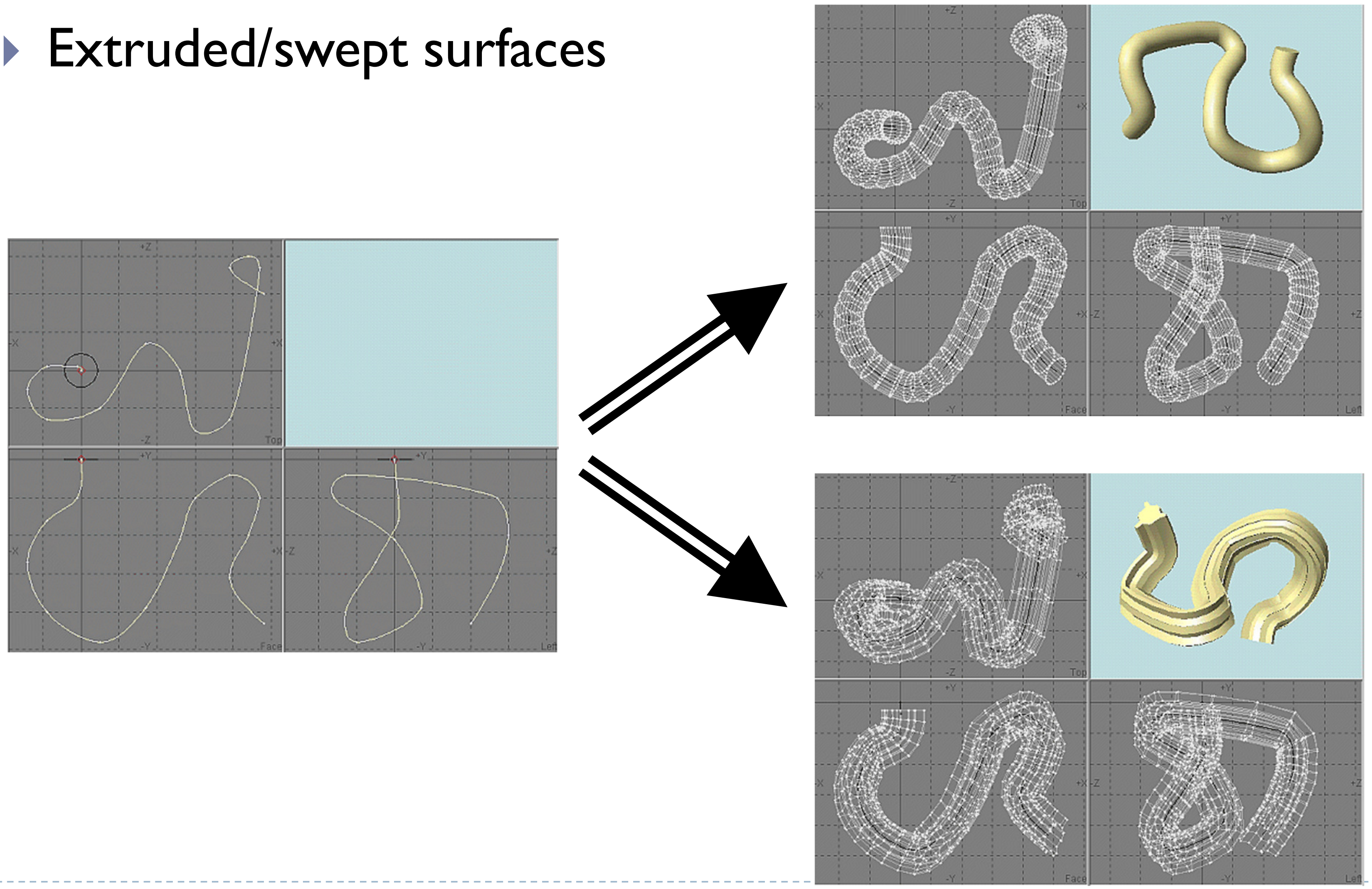
Curves

- Surface of revolution



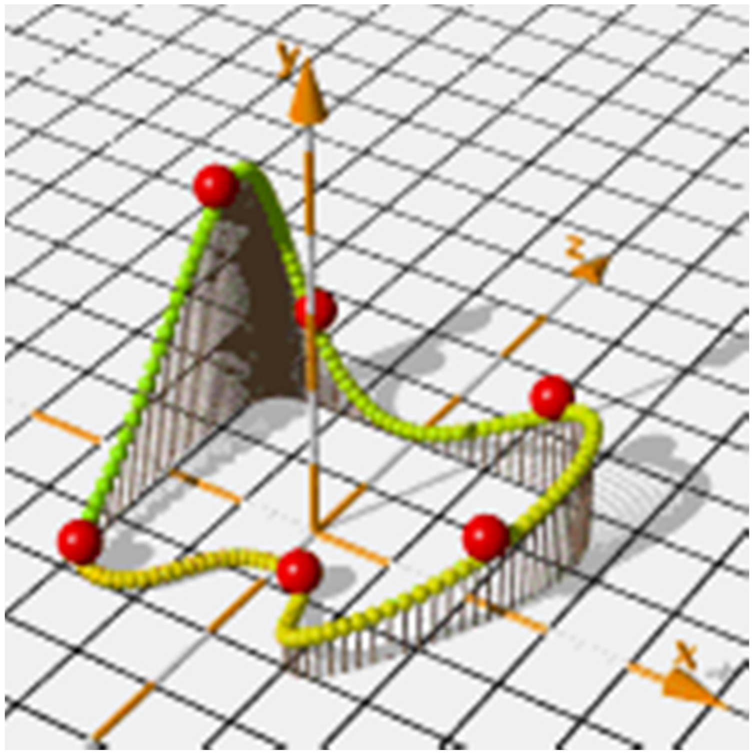
Curves

- Extruded/swept surfaces



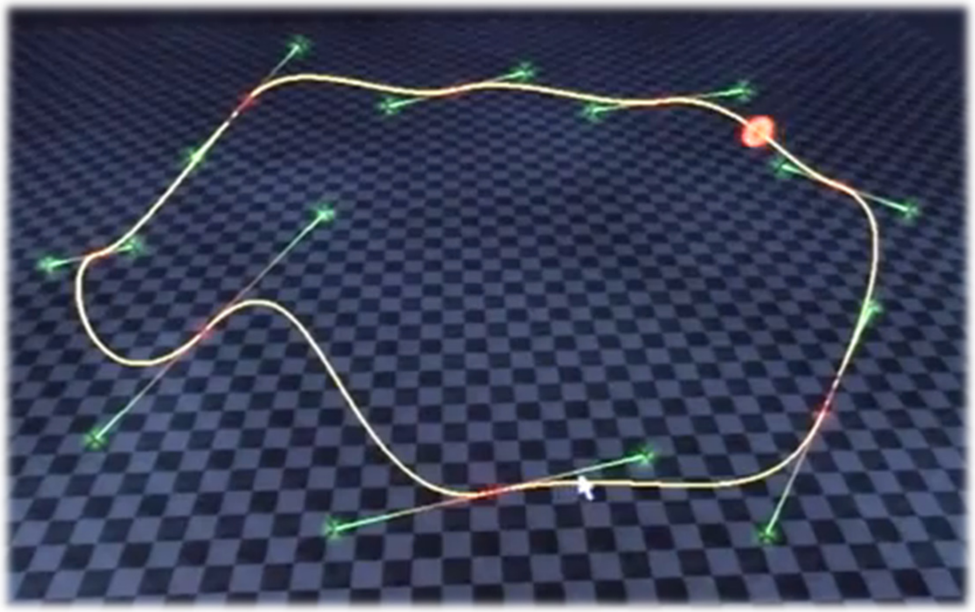
Curves

- Animation
  - Provide a "track" for objects
  - Use as camera path



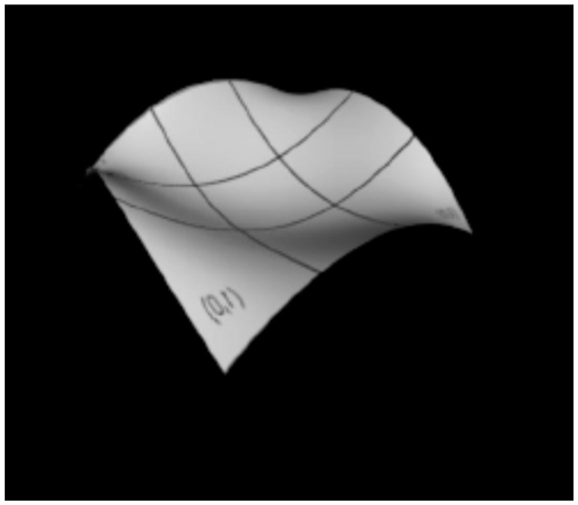
Video

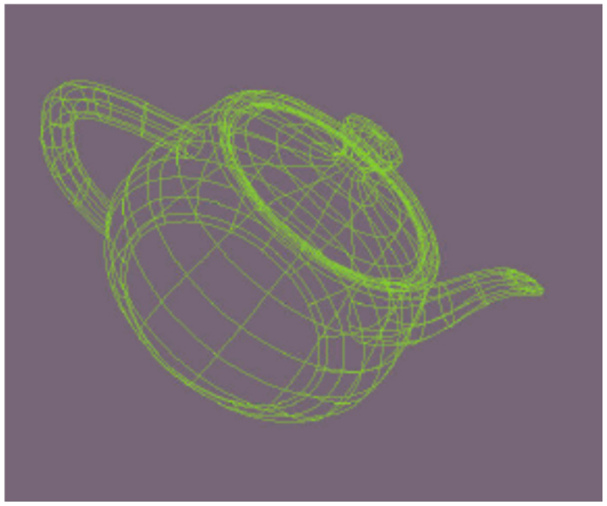
- Bezier Curves
  - http://www.youtube.com/watch?v=hlDYJNEiYvU



Curves

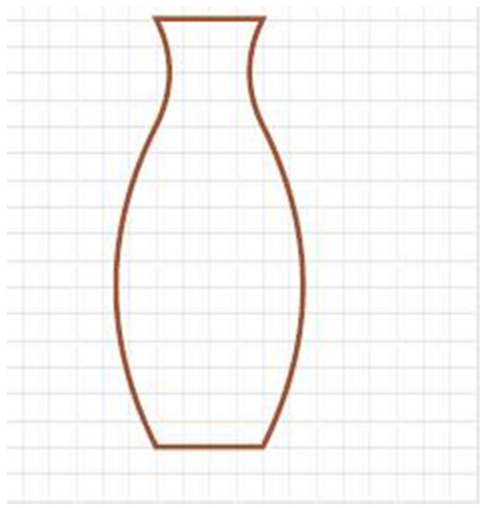
- Can be generalized to surface patches

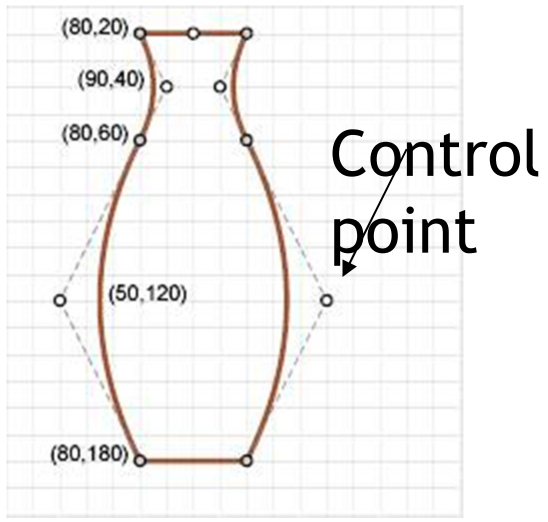




Curve Representation

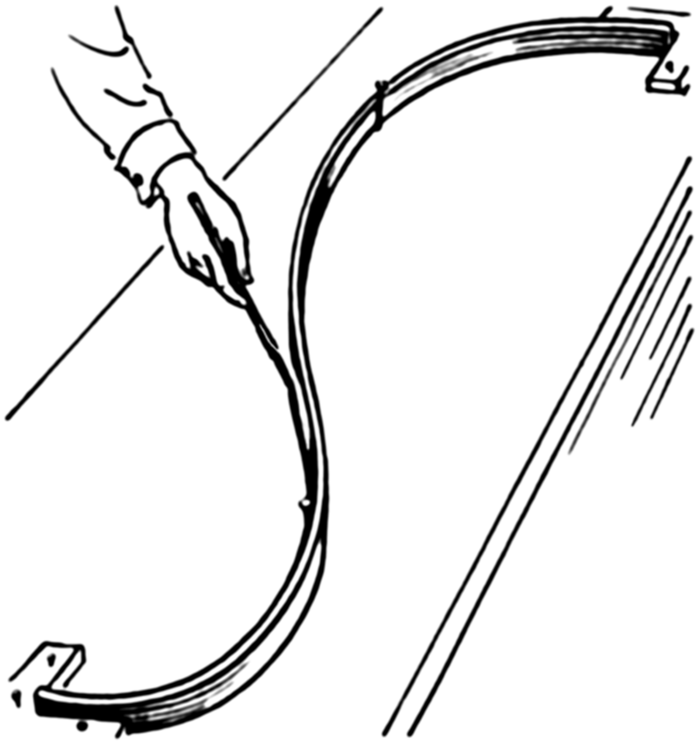
- Specify many points along a curve, connect with lines?
  - Difficult to get precise, smooth results across magnification levels
  - Large storage and CPU requirements
  - How many points are enough?
- Specify a curve using a small number of "control points"
  - Known as a *spline curve* or just *spline*





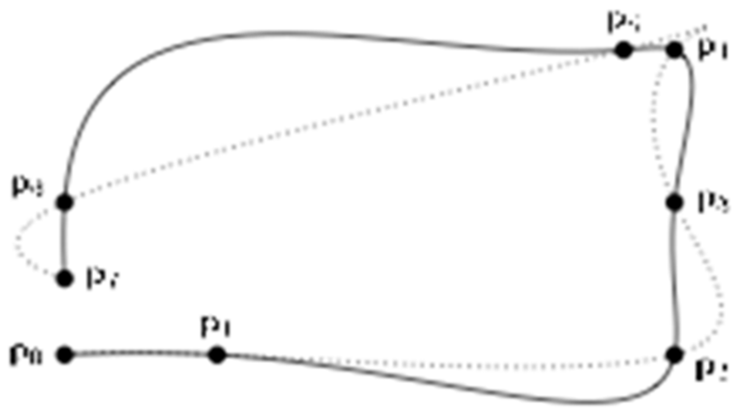
Spline: Definition

- Wikipedia:
  - Term comes from flexible spline devices used by shipbuilders and draftsmen to draw smooth shapes.
  - Spline consists of a long strip fixed in position at a number of points that relaxes to form a smooth curve passing through those points.



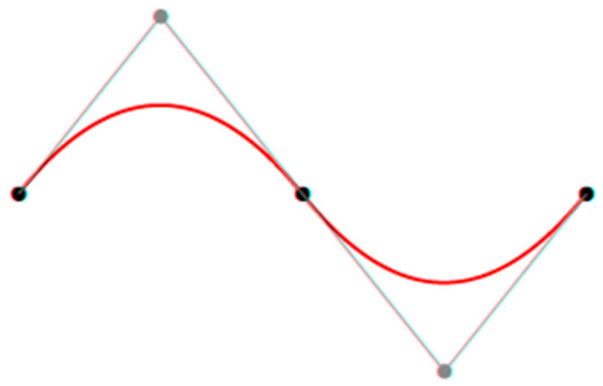
Interpolating Control Points

- "Interpolating" means that curve goes through all control points
- Seems most intuitive
- Surprisingly, not usually the best choice
  - Hard to predict behavior
  - Hard to get aesthetically pleasing curves



Approximating Control Points

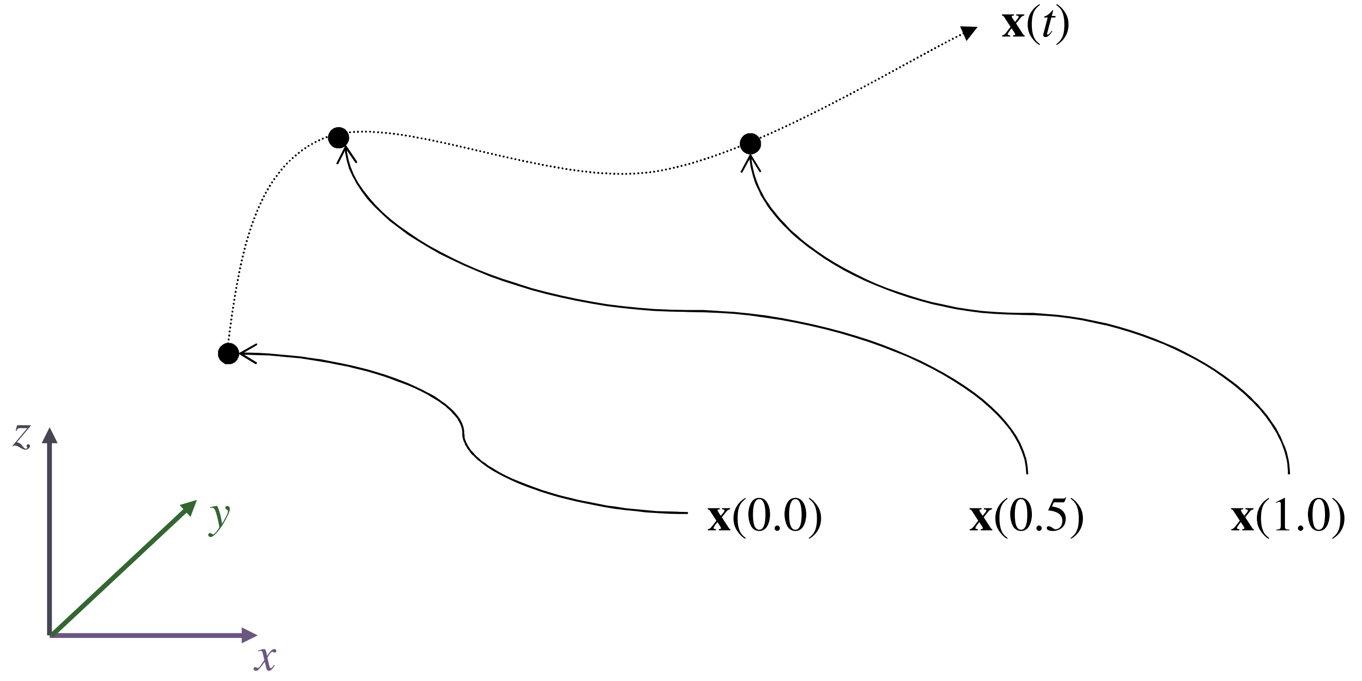
- Curve is "influenced" by control points



- Various types
- Most common: polynomial functions
  - Bézier spline (our focus)
  - B-spline (generalization of Bézier spline)
  - NURBS (Non Uniform Rational Basis Spline): used in CAD tools

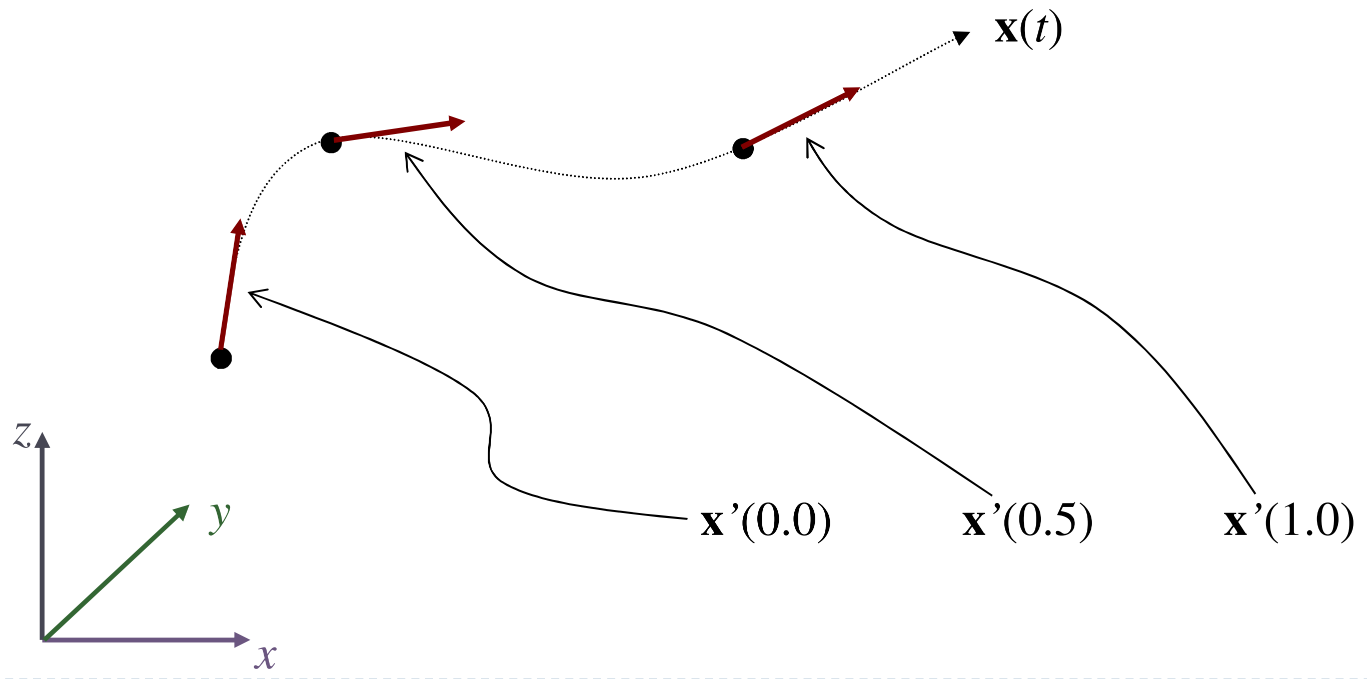
Mathematical Definition

- A vector valued function of one variable $\mathbf{x}(t)$
  - Given t, compute a 3D point $\mathbf{x}=(x,y,z)$
  - Could be interpreted as three functions: $x(t)$, $y(t)$, $z(t)$
  - Parameter t "moves a point along the curve"



Tangent Vector

- Derivative $\mathbf{x}'(t) = \frac{d\mathbf{x}}{dt} = (x'(t), y'(t), z'(t))$
- Vector x' points in direction of movement
- Length corresponds to speed



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  - Introduction
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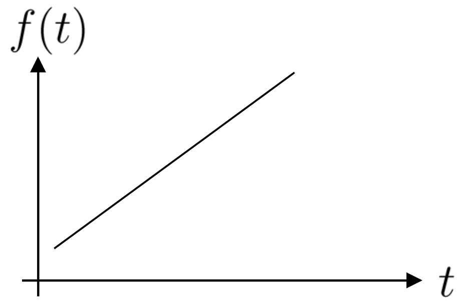
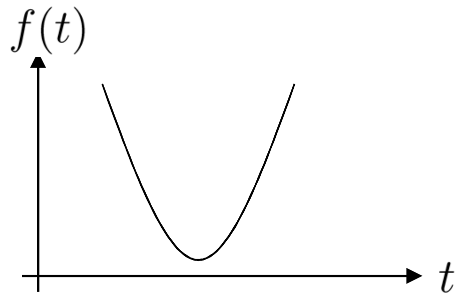
Polynomial Functions

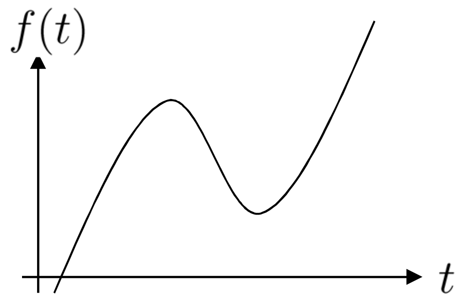
- Linear: (1st order) $$f(t) = at + b$$



- Quadratic: (2nd order) $$f(t) = at^2 + bt + c$$



- Cubic: (3rd order) $$f(t) = at^3 + bt^2 + ct + d$$



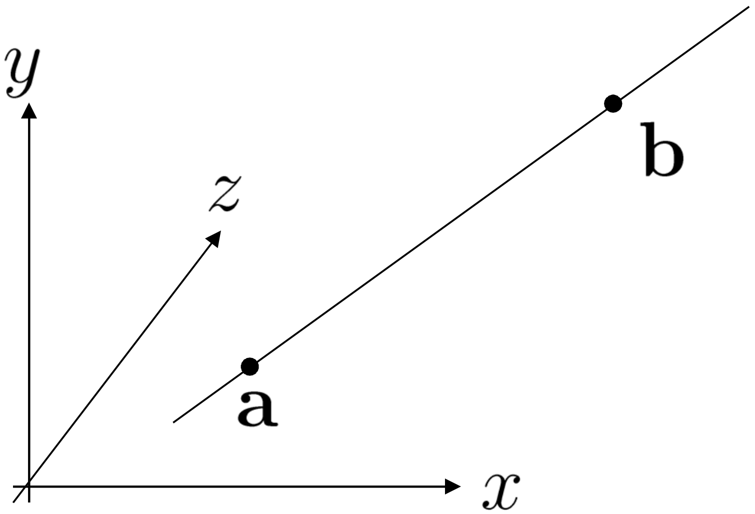
Polynomial Curves

- Linear $\mathbf{x}(t) = \mathbf{a}t + \mathbf{b}$

$$\mathbf{x} = (x, y, z), \mathbf{a} = (a_x, a_y, a_z), \mathbf{b} = (b_x, b_y, b_z)$$

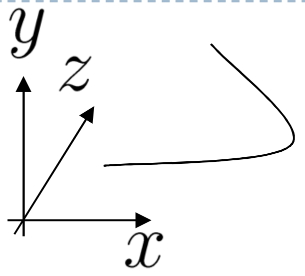
- Evaluated as:

$$x(t) = a_x t + b_x$$
$$y(t) = a_y t + b_y$$
$$z(t) = a_z t + b_z$$

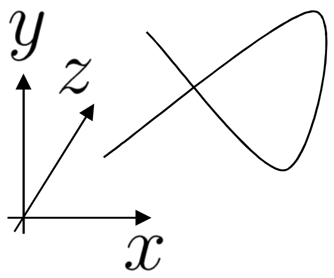


Polynomial Curves

- Quadratic: $\mathbf{x}(t) = \mathbf{a}t^2 + \mathbf{b}t + \mathbf{c}$
  (2nd order)



- Cubic: $\mathbf{x}(t) = \mathbf{a}t^3 + \mathbf{b}t^2 + \mathbf{c}t + \mathbf{d}$
  (3rd order)



- We usually define the curve for $0 \le t \le 1$

Control Points

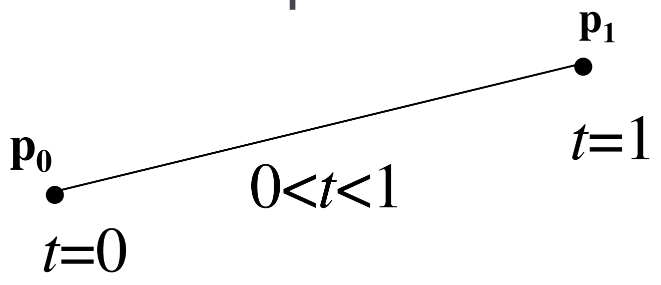
- Polynomial coefficients $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ can be interpreted as *control points*
  - Remember: $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ have x,y,z components each
- Unfortunately, they do not intuitively describe the shape of the curve
- Goal: intuitive control points

Control Points

- How many control points?
  - Two points define a line (1st order)
  - Three points define a quadratic curve (2nd order)
  - Four points define a cubic curve (3rd order)
  - $k+1$ points define a k-order curve
- Let’s start with a line…

First Order Curve

- Based on linear interpolation (LERP)
  - Weighted average between two values
  - “Value” could be a number, vector, color, …
- Interpolate between points $\mathbf{p}_0$ and $\mathbf{p}_1$ with parameter t
  - Defines a “curve” that is straight (first-order spline)
  - $t=0$ corresponds to $\mathbf{p}_0$
  - $t=1$ corresponds to $\mathbf{p}_1$
  - $t=0.5$ corresponds to midpoint



$$\mathbf{x}(t) = Lerp(t,\ \mathbf{p}_0,\ \mathbf{p}_1) = (1-t)\mathbf{p}_0 + t\ \mathbf{p}_1$$

Linear Interpolation

- Three equivalent ways to write it
  - Expose different properties

1. Regroup for points **p**

$$\mathbf{x}(t) = \mathbf{p}_0(1-t) + \mathbf{p}_1 t$$

2. Regroup for t

$$\mathbf{x}(t) = (\mathbf{p}_1 - \mathbf{p}_0)t + \mathbf{p}_0$$

3. Matrix form

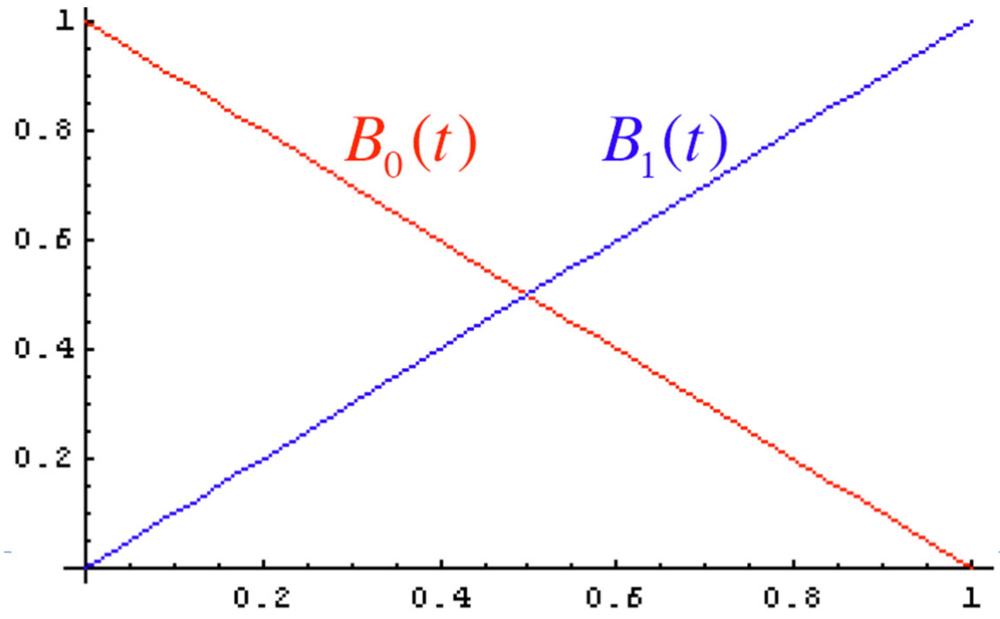
$$\mathbf{x}(t) = \begin{bmatrix} \mathbf{p}_0 & \mathbf{p}_1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} t \\ 1 \end{bmatrix}$$

Weighted Average

$$\mathbf{x}(t) = (1-t)\mathbf{p}_0 + \quad (t)\mathbf{p}_1$$

$$= B_0(t)\ \mathbf{p}_0 + B_1(t)\mathbf{p}_1, \text{ where } B_0(t) = 1-t \text{ and } B_1(t) = t$$

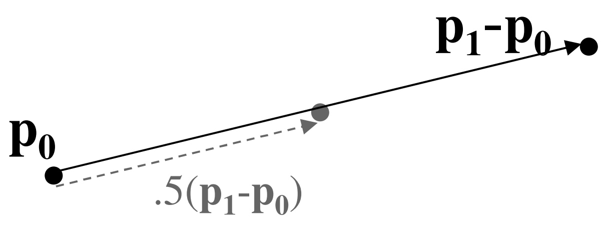
- Weights are a function of t
  - Sum is always 1, for any value of t
  - Also known as *blending functions*



Linear Polynomial

$$\mathbf{x}(t) = \underbrace{(\mathbf{p}_1 - \mathbf{p}_0)}_{\substack{\text{vector} \\ \mathbf{a}}}\, t + \underbrace{\mathbf{p}_0}_{\substack{\text{point} \\ \mathbf{b}}}$$

- Curve is based at point $\mathbf{p_0}$
- Add the vector, scaled by t


$\mathbf{p_1}$-$\mathbf{p_0}$
$\mathbf{p_0}$
.5($\mathbf{p_1}$-$\mathbf{p_0}$)

Matrix Form

$$\mathbf{x}(t) = \begin{bmatrix} \mathbf{p}_0 & \mathbf{p}_1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} t \\ 1 \end{bmatrix} = \mathbf{GBT}$$

- Geometry matrix $\mathbf{G} = \begin{bmatrix} \mathbf{p}_0 & \mathbf{p}_1 \end{bmatrix}$
- Geometric basis $\mathbf{B} = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$
- Polynomial basis $T = \begin{bmatrix} t \\ 1 \end{bmatrix}$
- In components

$$\mathbf{x}(t) = \begin{bmatrix} p_{0x} & p_{1x} \\ p_{0y} & p_{1y} \\ p_{0z} & p_{1z} \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} t \\ 1 \end{bmatrix}$$

Tangent

- For a straight line, the tangent is constant

$$\mathbf{x}'(t) = \mathbf{p}_1 - \mathbf{p}_0$$

- Weighted average $\mathbf{x}'(t) = (-1)\mathbf{p}_0 + (+1)\mathbf{p}_1$

- Polynomial $\mathbf{x}'(t) = 0t + (\mathbf{p}_1 - \mathbf{p}_0)$

- Matrix form $\mathbf{x}'(t) = \begin{bmatrix} \mathbf{p}_0 & \mathbf{p}_1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

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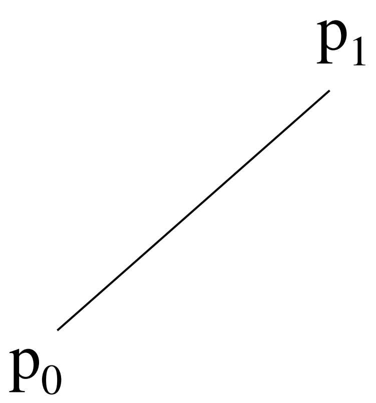
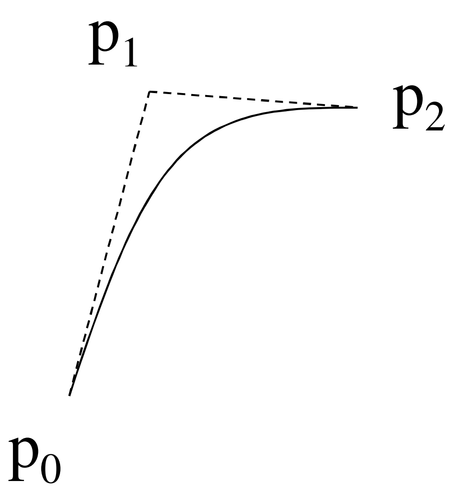
Bézier Curves

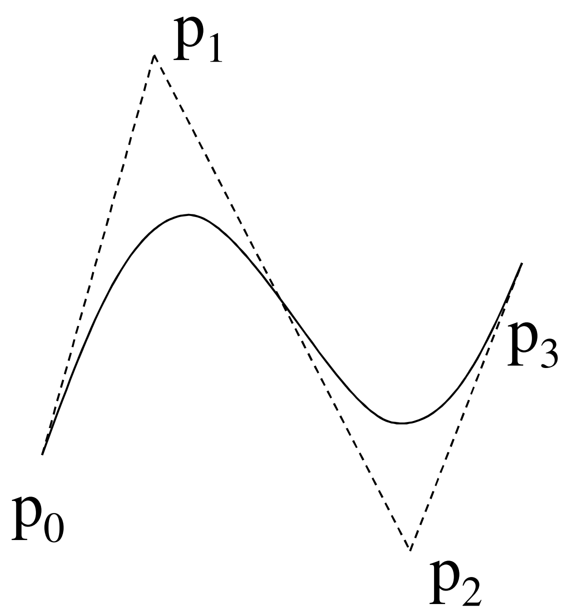
- Are a higher order extension of linear interpolation



Linear



Quadratic



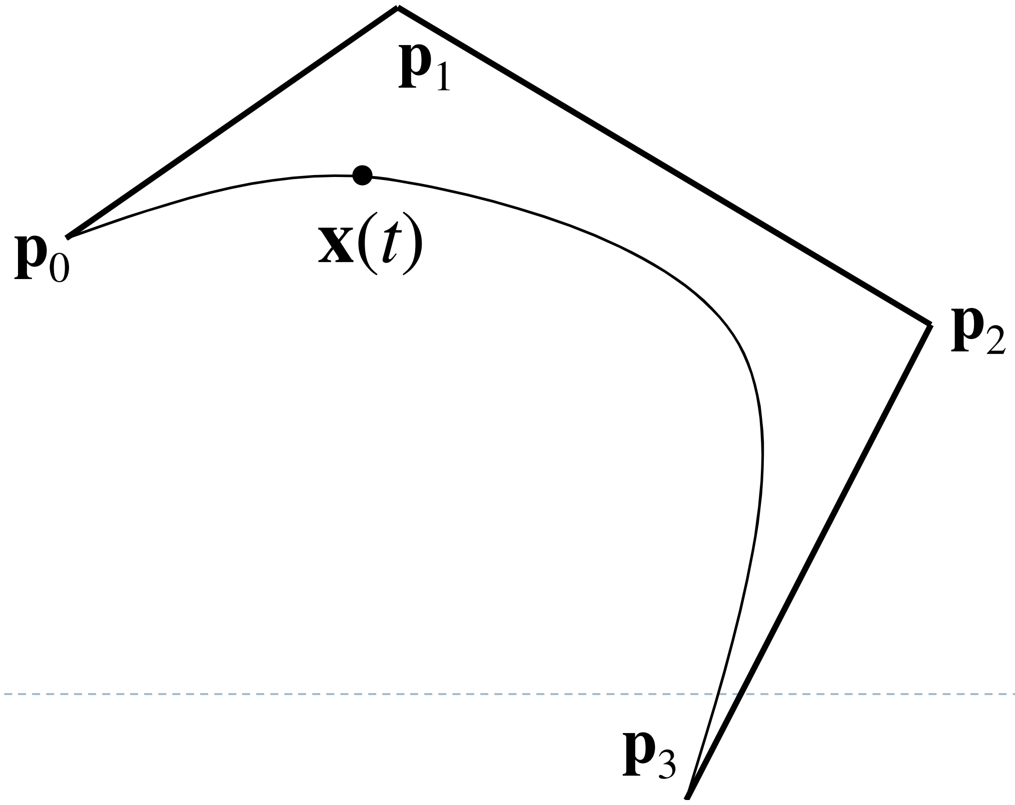
Cubic

Bézier Curves

- Give intuitive control over curve with control points
  - Endpoints are interpolated, intermediate points are approximated
  - Convex Hull property
- Many demo applets online, for example:
  - Demo: http://www.cs.princeton.edu/~min/cs426/jar/bezier.html
  - http://www.theparticle.com/applets/nyu/BezierApplet/
  - http://www.sunsite.ubc.ca/LivingMathematics/V001N01/UBCExamples/Bezier/bezier.html

Cubic Bézier Curve

- Most commonly used case
- Defined by four control points:
  - Two interpolated endpoints (points are on the curve)
  - Two points control the tangents at the endpoints
- Points $\mathbf{x}$ on curve defined as function of parameter t



Algorithmic Construction

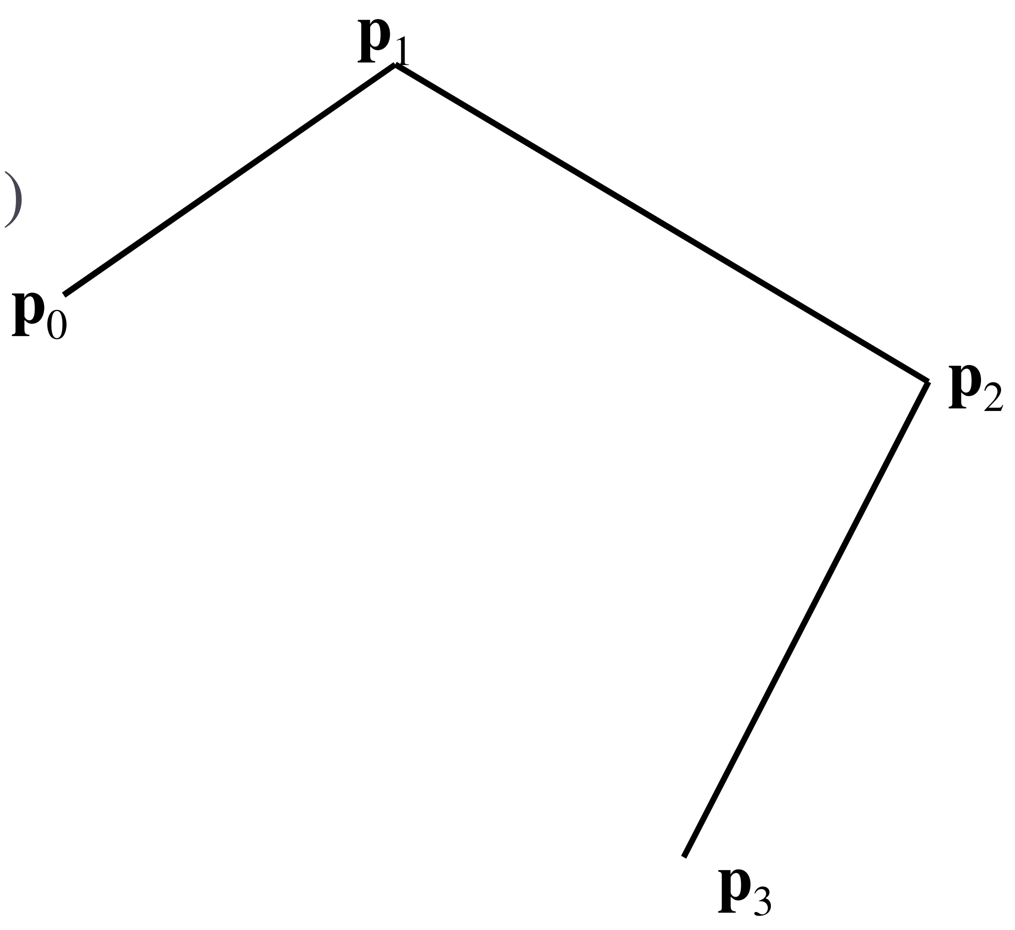
- Algorithmic construction
  - *De Casteljau* algorithm, developed at Citroen in 1959, named after its inventor Paul de Casteljau (pronounced “Cast-all-’Joe”)
  - Developed independently from Bézier’s work:
    Bézier created the formulation using blending functions,
    Casteljau devised the recursive interpolation algorithm

De Casteljau Algorithm

- A recursive series of linear interpolations
  - Works for any order Bezier function, not only cubic
- Not very efficient to evaluate
  - Other forms more commonly used
- But:
  - Gives intuition about the geometry
  - Useful for subdivision

De Casteljau Algorithm

- Given:
  - Four control points
  - A value of t (here $t \approx 0.25$)

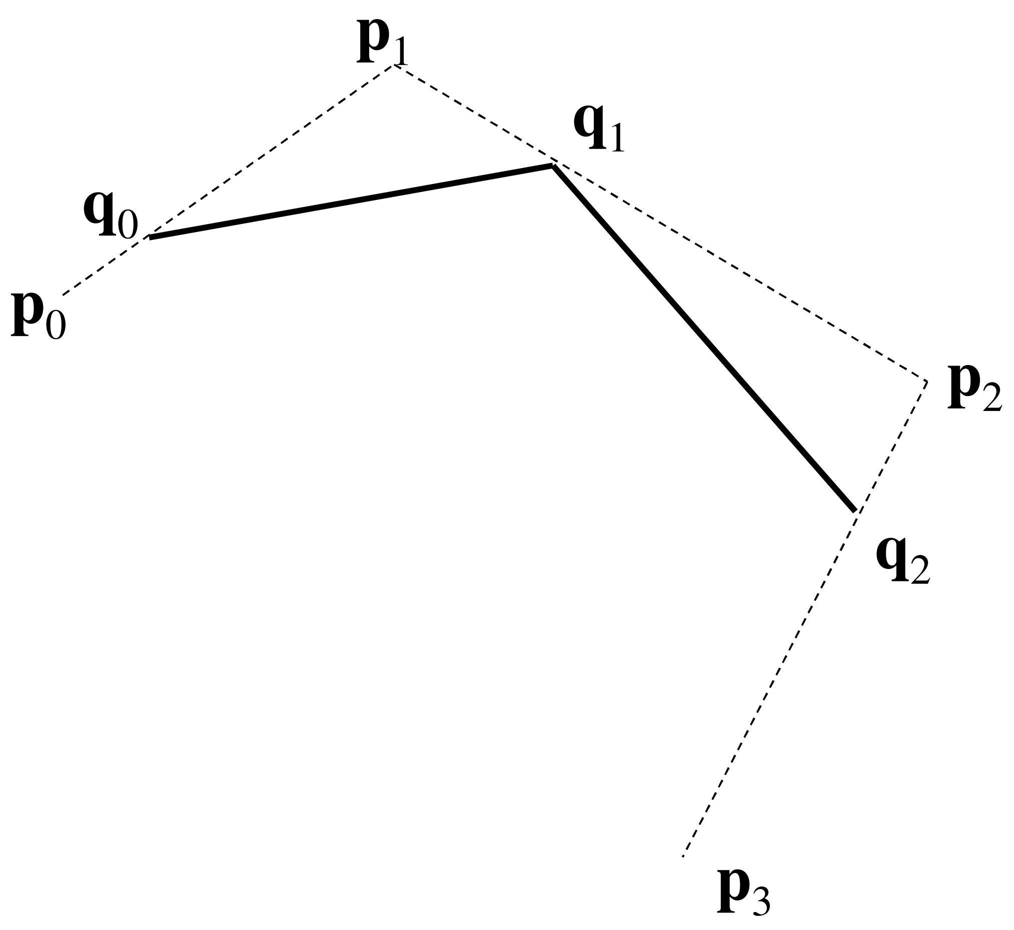


De Casteljau Algorithm

$$\mathbf{q}_0(t) = Lerp\left(t, \mathbf{p}_0, \mathbf{p}_1\right)$$

$$\mathbf{q}_1(t) = Lerp\left(t, \mathbf{p}_1, \mathbf{p}_2\right)$$

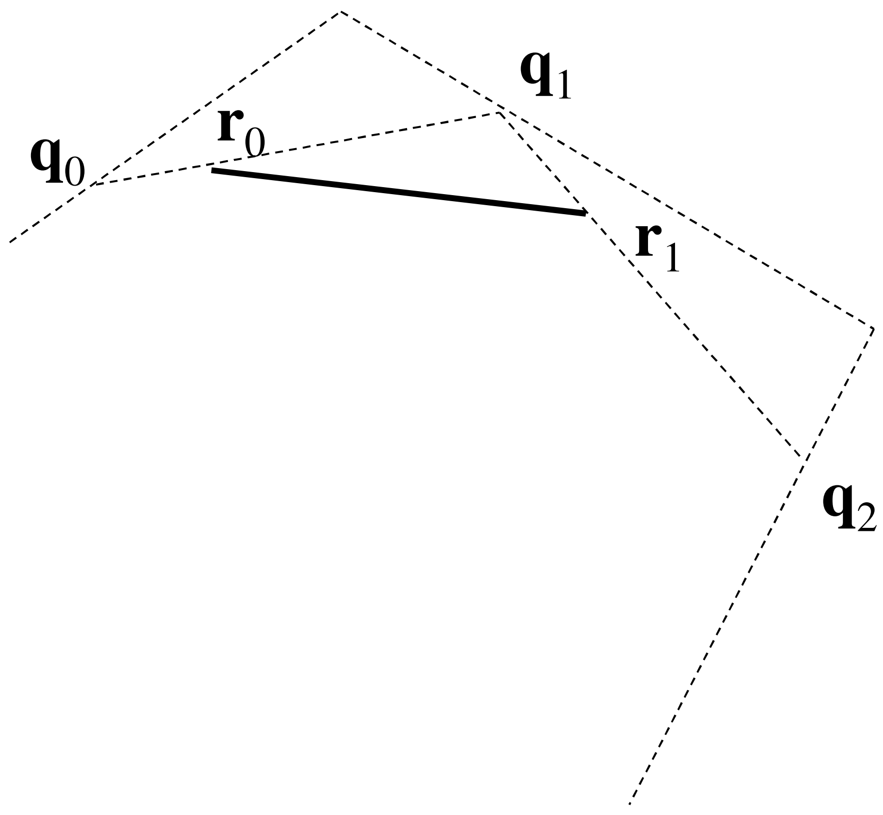
$$\mathbf{q}_2(t) = Lerp\left(t, \mathbf{p}_2, \mathbf{p}_3\right)$$



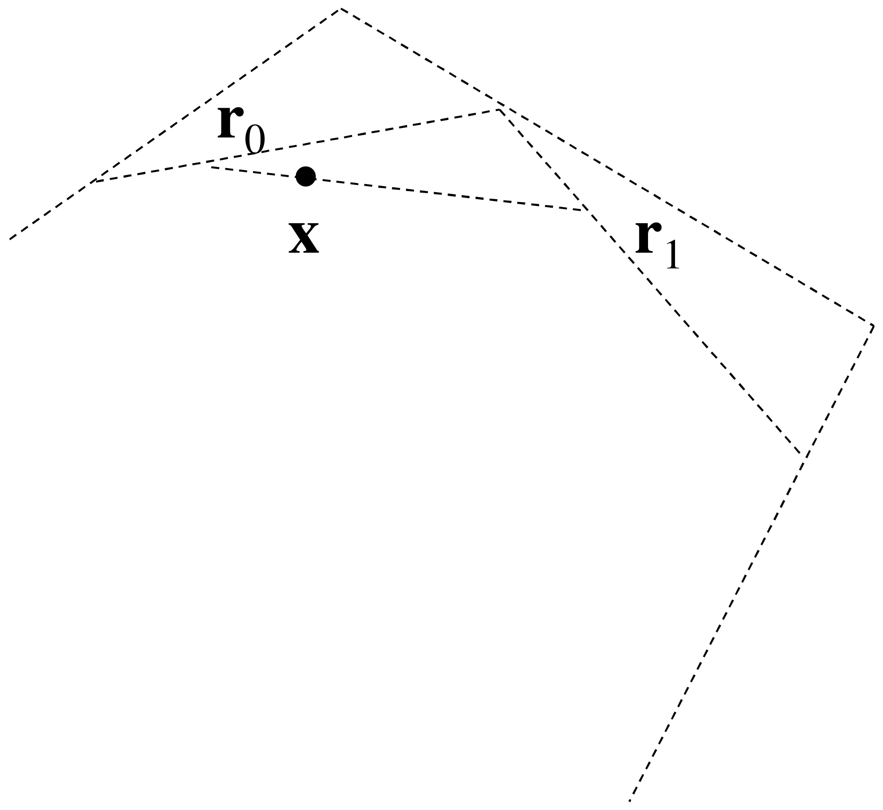
De Casteljau Algorithm

$$\mathbf{r}_0(t) = Lerp\big(t, \mathbf{q}_0(t), \mathbf{q}_1(t)\big)$$

$$\mathbf{r}_1(t) = Lerp\big(t, \mathbf{q}_1(t), \mathbf{q}_2(t)\big)$$

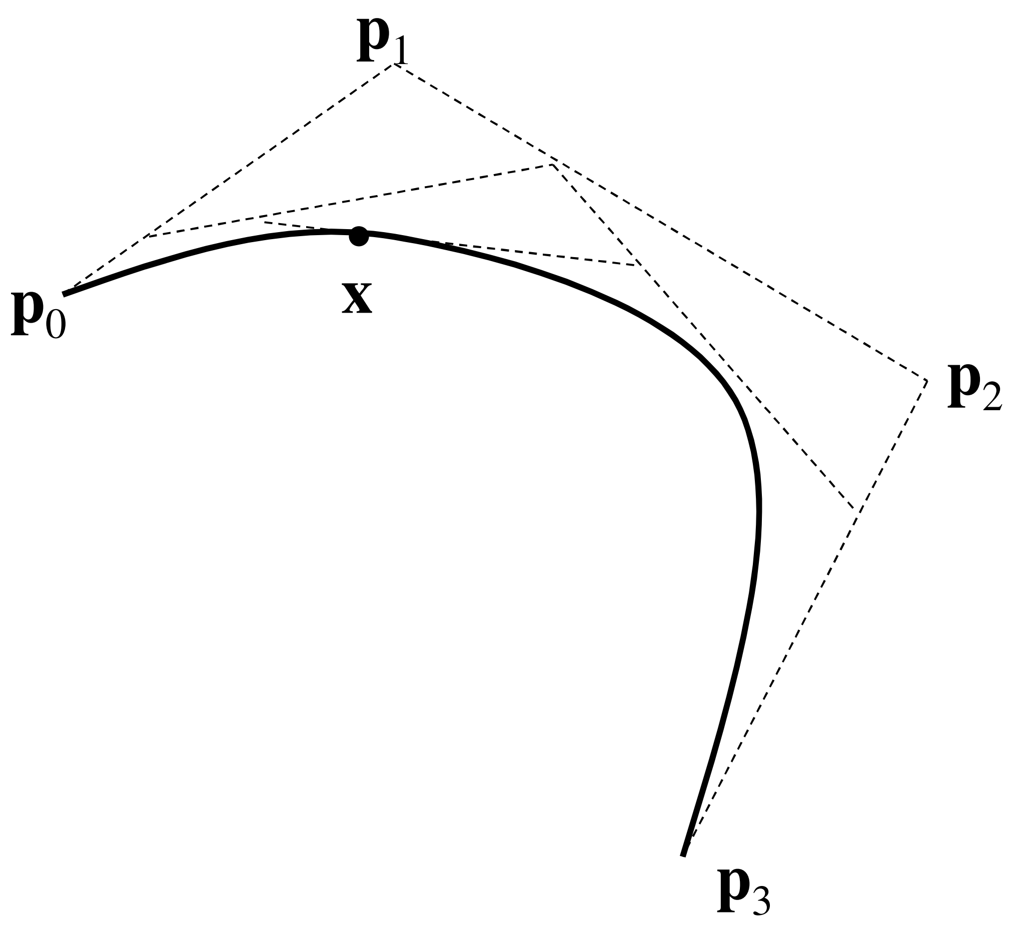


De Casteljau Algorithm



$$\mathbf{x}(t) = Lerp\left(t, \mathbf{r}_0(t), \mathbf{r}_1(t)\right)$$

De Casteljau Algorithm



- Applets
  - Demo: http://www2.mat.dtu.dk/people/J.Gravesen/cagd/decast.html
  - http://www.caffeineowl.com/graphics/2d/vectorial/bezierintro.html

Recursive Linear Interpolation

$$\mathbf{x} = Lerp(t, \mathbf{r}_0, \mathbf{r}_1)$$

$$\mathbf{r}_0 = Lerp(t, \mathbf{q}_0, \mathbf{q}_1)$$
$$\mathbf{r}_1 = Lerp(t, \mathbf{q}_1, \mathbf{q}_2)$$

$$\mathbf{q}_0 = Lerp(t, \mathbf{p}_0, \mathbf{p}_1)$$
$$\mathbf{q}_1 = Lerp(t, \mathbf{p}_1, \mathbf{p}_2)$$
$$\mathbf{q}_2 = Lerp(t, \mathbf{p}_2, \mathbf{p}_3)$$

$$\mathbf{p}_0$$
$$\mathbf{p}_1$$
$$\mathbf{p}_2$$
$$\mathbf{p}_3$$

$$\begin{tikzcd}
 & & & \mathbf{p}_1 \arrow[dl] \\
 & & \mathbf{q}_0 \arrow[dl] & \\
 & \mathbf{r}_0 \arrow[dl] & & \mathbf{p}_2 \arrow[ul] \arrow[dl] \\
\mathbf{x} & & \mathbf{q}_1 \arrow[ul] \arrow[dl] & \\
 & \mathbf{r}_1 \arrow[ul] & & \mathbf{p}_3 \arrow[ul] \arrow[dl] \\
 & & \mathbf{q}_2 \arrow[ul] & \\
 & & & \mathbf{p}_4 \arrow[ul]
\end{tikzcd}$$

Expand the LERPs

$$\mathbf{q}_0(t) = Lerp\left(t, \mathbf{p}_0, \mathbf{p}_1\right) = \left(1-t\right)\mathbf{p}_0 + t\mathbf{p}_1$$

$$\mathbf{q}_1(t) = Lerp\left(t, \mathbf{p}_1, \mathbf{p}_2\right) = \left(1-t\right)\mathbf{p}_1 + t\mathbf{p}_2$$

$$\mathbf{q}_2(t) = Lerp\left(t, \mathbf{p}_2, \mathbf{p}_3\right) = \left(1-t\right)\mathbf{p}_2 + t\mathbf{p}_3$$

$$\mathbf{r}_0(t) = Lerp\left(t, \mathbf{q}_0(t), \mathbf{q}_1(t)\right) = \left(1-t\right)\left(\left(1-t\right)\mathbf{p}_0 + t\mathbf{p}_1\right) + t\left(\left(1-t\right)\mathbf{p}_1 + t\mathbf{p}_2\right)$$

$$\mathbf{r}_1(t) = Lerp\left(t, \mathbf{q}_1(t), \mathbf{q}_2(t)\right) = \left(1-t\right)\left(\left(1-t\right)\mathbf{p}_1 + t\mathbf{p}_2\right) + t\left(\left(1-t\right)\mathbf{p}_2 + t\mathbf{p}_3\right)$$

$$\begin{aligned}
\mathbf{x}(t) &= Lerp\left(t, \mathbf{r}_0(t), \mathbf{r}_1(t)\right) \\
&= \left(1-t\right)\left(\left(1-t\right)\left(\left(1-t\right)\mathbf{p}_0 + t\mathbf{p}_1\right) + t\left(\left(1-t\right)\mathbf{p}_1 + t\mathbf{p}_2\right)\right) \\
&\qquad + t\left(\left(1-t\right)\left(\left(1-t\right)\mathbf{p}_1 + t\mathbf{p}_2\right) + t\left(\left(1-t\right)\mathbf{p}_2 + t\mathbf{p}_3\right)\right)
\end{aligned}$$

Weighted Average of Control Points

- Regroup for p:

$$\mathbf{x}(t) = (1-t)\Big((1-t)\big((1-t)\mathbf{p}_0 + t\mathbf{p}_1\big) + t\big((1-t)\mathbf{p}_1 + t\mathbf{p}_2\big)\Big)$$
$$+t\Big((1-t)\big((1-t)\mathbf{p}_1 + t\mathbf{p}_2\big) + t\big((1-t)\mathbf{p}_2 + t\mathbf{p}_3\big)\Big)$$

$$\mathbf{x}(t) = (1-t)^3\,\mathbf{p}_0 + 3(1-t)^2\,t\mathbf{p}_1 + 3(1-t)t^2\mathbf{p}_2 + t^3\mathbf{p}_3$$

$$\mathbf{x}(t) = \overbrace{\left(-t^3 + 3t^2 - 3t + 1\right)}^{B_0(t)}\mathbf{p}_0 + \overbrace{\left(3t^3 - 6t^2 + 3t\right)}^{B_1(t)}\mathbf{p}_1$$
$$+\underbrace{\left(-3t^3 + 3t^2\right)}_{B_2(t)}\mathbf{p}_2 + \underbrace{\left(t^3\right)}_{B_3(t)}\mathbf{p}_3$$

Cubic Bernstein Polynomials

$$\mathbf{x}(t) = B_0(t)\mathbf{p}_0 + B_1(t)\mathbf{p}_1 + B_2(t)\mathbf{p}_2 + B_3(t)\mathbf{p}_3$$

The cubic *Bernstein polynomials :*

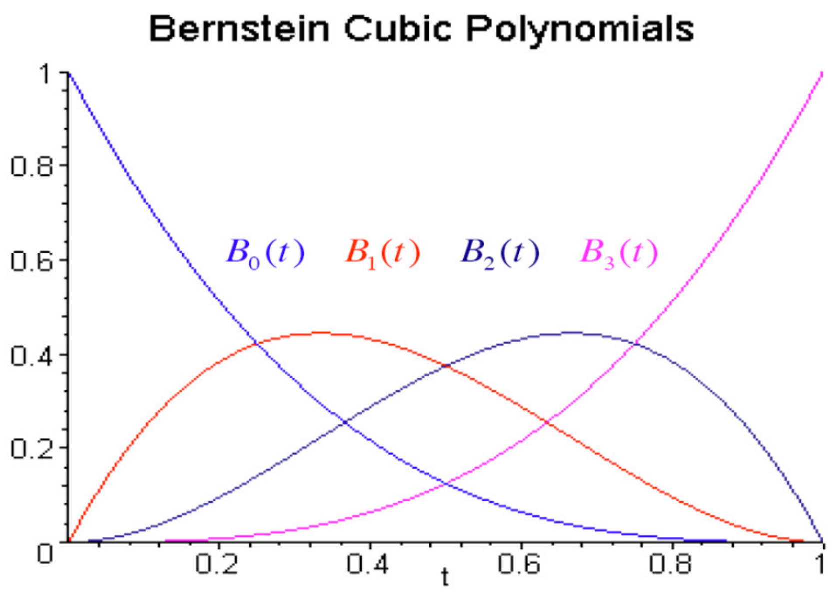
$$B_0(t) = -t^3 + 3t^2 - 3t + 1$$

$$B_1(t) = 3t^3 - 6t^2 + 3t$$

$$B_2(t) = -3t^3 + 3t^2$$

$$B_3(t) = t^3$$

$$\sum B_i(t) = 1$$



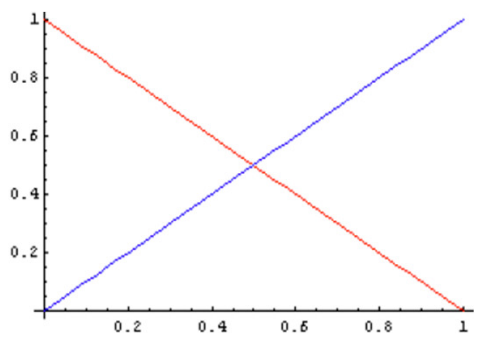
- Weights $B_i(t)$ add up to 1 for any value of t

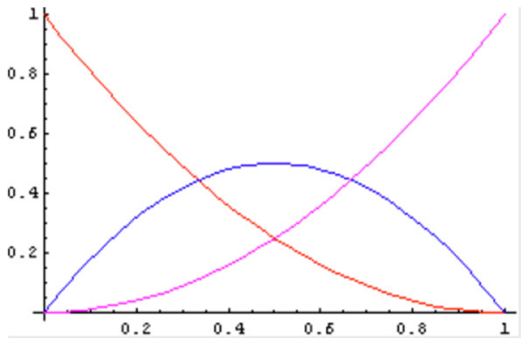
General Bernstein Polynomials

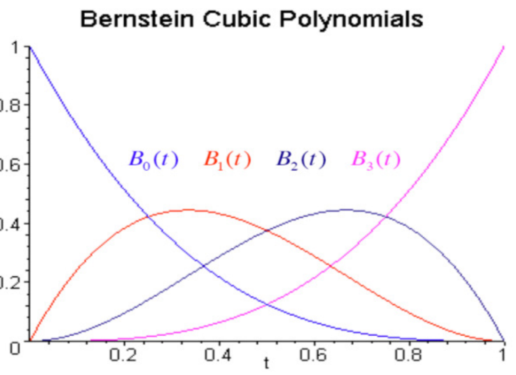
$$B_0^1(t) = -t + 1$$
$$B_1^1(t) = t$$

$$B_0^2(t) = t^2 - 2t + 1$$
$$B_1^2(t) = -2t^2 + 2t$$
$$B_2^2(t) = t^2$$

$$B_0^3(t) = -t^3 + 3t^2 - 3t + 1$$
$$B_1^3(t) = 3t^3 - 6t^2 + 3t$$
$$B_2^3(t) = -3t^3 + 3t^2$$
$$B_3^3(t) = t^3$$







$$B_i^n(t) = \binom{n}{i} (1-t)^{n-i} (t)^i \qquad \binom{n}{i} = \frac{n!}{i!(n-i)!}$$

$$\sum B_i^n(t) = 1$$

n! = factorial of n
(n+1)! = n! x (n+1)

General Bézier Curves

- nth-order Bernstein polynomials form nth-order Bézier curves

$$B_i^n(t) = \binom{n}{i}(1-t)^{n-i}(t)^i$$

$$\mathbf{x}(t) = \sum_{i=0}^{n} B_i^n(t)\mathbf{p}_i$$

Bézier Curve Properties

Overview:

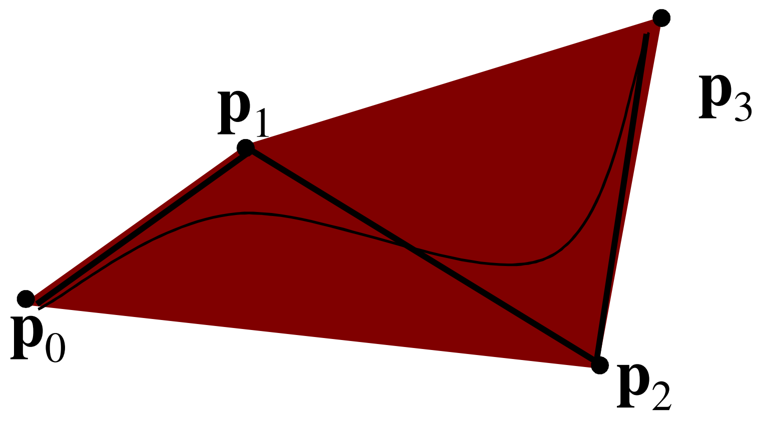
- Convex Hull property
- Affine Invariance

Definitions

- Convex hull of a set of points:
  - Polyhedral volume created such that all lines connecting any two points lie completely inside it (or on its boundary)
- Convex combination of a set of points:
  - Weighted average of the points, where all weights between 0 and 1, sum up to 1
- Any convex combination of a set of points lies within the convex hull

Convex Hull Property

- A Bézier curve is a convex combination of the control points (by definition, see Bernstein polynomials)
- A Bézier curve is always inside the convex hull
  - Makes curve predictable
  - Allows culling, intersection testing, adaptive tessellation
- Demo: http://www.cs.princeton.edu/~min/cs426/jar/bezier.html



Affine Invariance

Transforming Bézier curves

- Two ways to transform:
  - Transform the control points, then compute resulting spline points
  - Compute spline points, then transform them
- Either way, we get the same points
  - Curve is defined via affine combination of points
  - Invariant under affine transformations (i.e., translation, scale, rotation, shear)
  - Convex hull property remains true

Cubic Polynomial Form

Start with Bernstein form:

$$\mathbf{x}(t) = \left(-t^3 + 3t^2 - 3t + 1\right)\mathbf{p}_0 + \left(3t^3 - 6t^2 + 3t\right)\mathbf{p}_1 + \left(-3t^3 + 3t^2\right)\mathbf{p}_2 + \left(t^3\right)\mathbf{p}_3$$

Regroup into coefficients of t :

$$\mathbf{x}(t) = \left(-\mathbf{p}_0 + 3\mathbf{p}_1 - 3\mathbf{p}_2 + \mathbf{p}_3\right)t^3 + \left(3\mathbf{p}_0 - 6\mathbf{p}_1 + 3\mathbf{p}_2\right)t^2 + \left(-3\mathbf{p}_0 + 3\mathbf{p}_1\right)t + \left(\mathbf{p}_0\right)1$$

$$\boxed{\mathbf{x}(t) = \mathbf{a}t^3 + \mathbf{b}t^2 + \mathbf{c}t + \mathbf{d} \qquad \begin{aligned} \mathbf{a} &= \left(-\mathbf{p}_0 + 3\mathbf{p}_1 - 3\mathbf{p}_2 + \mathbf{p}_3\right) \\ \mathbf{b} &= \left(3\mathbf{p}_0 - 6\mathbf{p}_1 + 3\mathbf{p}_2\right) \\ \mathbf{c} &= \left(-3\mathbf{p}_0 + 3\mathbf{p}_1\right) \\ \mathbf{d} &= \left(\mathbf{p}_0\right) \end{aligned}}$$

- Good for fast evaluation
  - Precompute constant coefficients (**a**,**b**,**c**,**d**)
- Not much geometric intuition

Cubic Matrix Form

$$\mathbf{x}(t) = \begin{bmatrix} \vec{\mathbf{a}} & \vec{\mathbf{b}} & \vec{\mathbf{c}} & \mathbf{d} \end{bmatrix} \begin{bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}$$

$$\vec{\mathbf{a}} = \left(-\mathbf{p}_0 + 3\mathbf{p}_1 - 3\mathbf{p}_2 + \mathbf{p}_3\right)$$
$$\vec{\mathbf{b}} = \left(3\mathbf{p}_0 - 6\mathbf{p}_1 + 3\mathbf{p}_2\right)$$
$$\vec{\mathbf{c}} = \left(-3\mathbf{p}_0 + 3\mathbf{p}_1\right)$$
$$\mathbf{d} = \left(\mathbf{p}_0\right)$$

$$\mathbf{x}(t) = \underbrace{\begin{bmatrix} \mathbf{p}_0 & \mathbf{p}_1 & \mathbf{p}_2 & \mathbf{p}_3 \end{bmatrix}}_{\mathbf{G}_{Bez}} \underbrace{\begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}}_{\mathbf{B}_{Bez}} \underbrace{\begin{bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}}_{\mathbf{T}}$$

- Other types of cubic splines use different basis matrices $\mathbf{B}_{\mathrm{Bez}}$

Cubic Matrix Form

- In 3D: 3 equations for x, y and z:

$$\mathbf{x}_x(t)=\begin{bmatrix}p_{0x} & p_{1x} & p_{2x} & p_{3x}\end{bmatrix}\begin{bmatrix}-1 & 3 & -3 & 1\\ 3 & -6 & 3 & 0\\ -3 & 3 & 0 & 0\\ 1 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}t^3\\ t^2\\ t\\ 1\end{bmatrix}$$

$$\mathbf{x}_y(t)=\begin{bmatrix}p_{0y} & p_{1y} & p_{2y} & p_{3y}\end{bmatrix}\begin{bmatrix}-1 & 3 & -3 & 1\\ 3 & -6 & 3 & 0\\ -3 & 3 & 0 & 0\\ 1 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}t^3\\ t^2\\ t\\ 1\end{bmatrix}$$

$$\mathbf{x}_z(t)=\begin{bmatrix}p_{0z} & p_{1z} & p_{2z} & p_{3z}\end{bmatrix}\begin{bmatrix}-1 & 3 & -3 & 1\\ 3 & -6 & 3 & 0\\ -3 & 3 & 0 & 0\\ 1 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}t^3\\ t^2\\ t\\ 1\end{bmatrix}$$

Matrix Form

- Bundle into a single matrix

$$\mathbf{x}(t) = \begin{bmatrix} p_{0x} & p_{1x} & p_{2x} & p_{3x} \\ p_{0y} & p_{1y} & p_{2y} & p_{3y} \\ p_{0z} & p_{1z} & p_{2z} & p_{3z} \end{bmatrix} \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}$$

$$\boxed{\begin{aligned} \mathbf{x}(t) &= \mathbf{G}_{Bez}\mathbf{B}_{Bez}\mathbf{T} \\ \mathbf{x}(t) &= \mathbf{C}\ \ \mathbf{T} \end{aligned}}$$

- Efficient evaluation
  - Pre-compute $\mathbf{C}$
  - Take advantage of existing 4x4 matrix hardware support

Lecture Overview

- Polynomial Curves
  - Introduction
  - Polynomial functions
- Bézier Curves
  - Introduction
  - Drawing Bézier curves
  - Piecewise Bézier curves

Drawing Bézier Curves

- Draw *line segments* or individual pixels
- Approximate the curve as a series of line segments (*tessellation*)
  - Uniform sampling
  - Adaptive sampling
  - Recursive subdivision

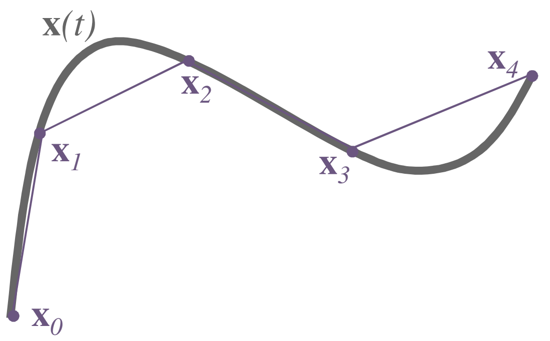
Uniform Sampling

- Approximate curve with N straight segments
  - N chosen in advance
  - Evaluate

$$\mathbf{x}_i = \mathbf{x}(t_i) \text{ where } t_i = \frac{i}{N} \text{ for } i = 0, 1, \ldots, N$$

$$\mathbf{x}_i = \vec{\mathbf{a}}\frac{i^3}{N^3} + \vec{\mathbf{b}}\frac{i^2}{N^2} + \vec{\mathbf{c}}\frac{i}{N} + \mathbf{d}$$

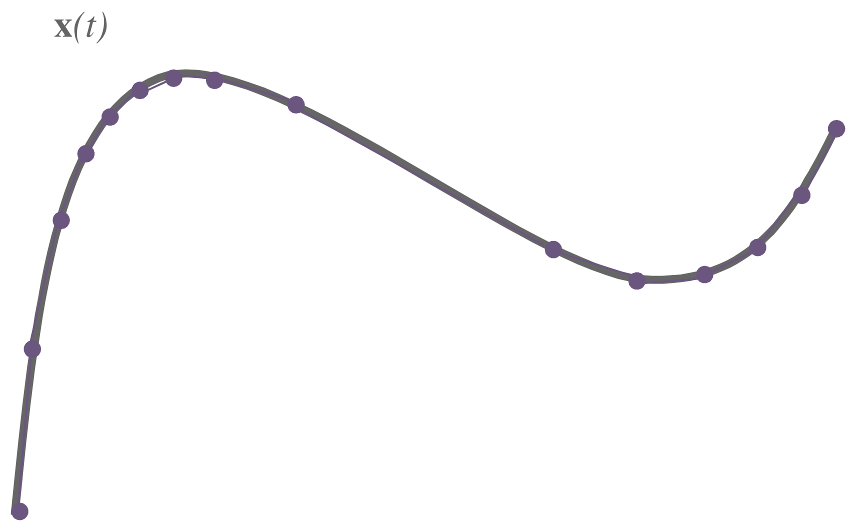
  - Connect the points with lines



- Too few points?
  - Poor approximation
  - “Curve” is faceted
- Too many points?
  - Slow to draw too many line segments
  - Segments may draw on top of each other

Adaptive Sampling

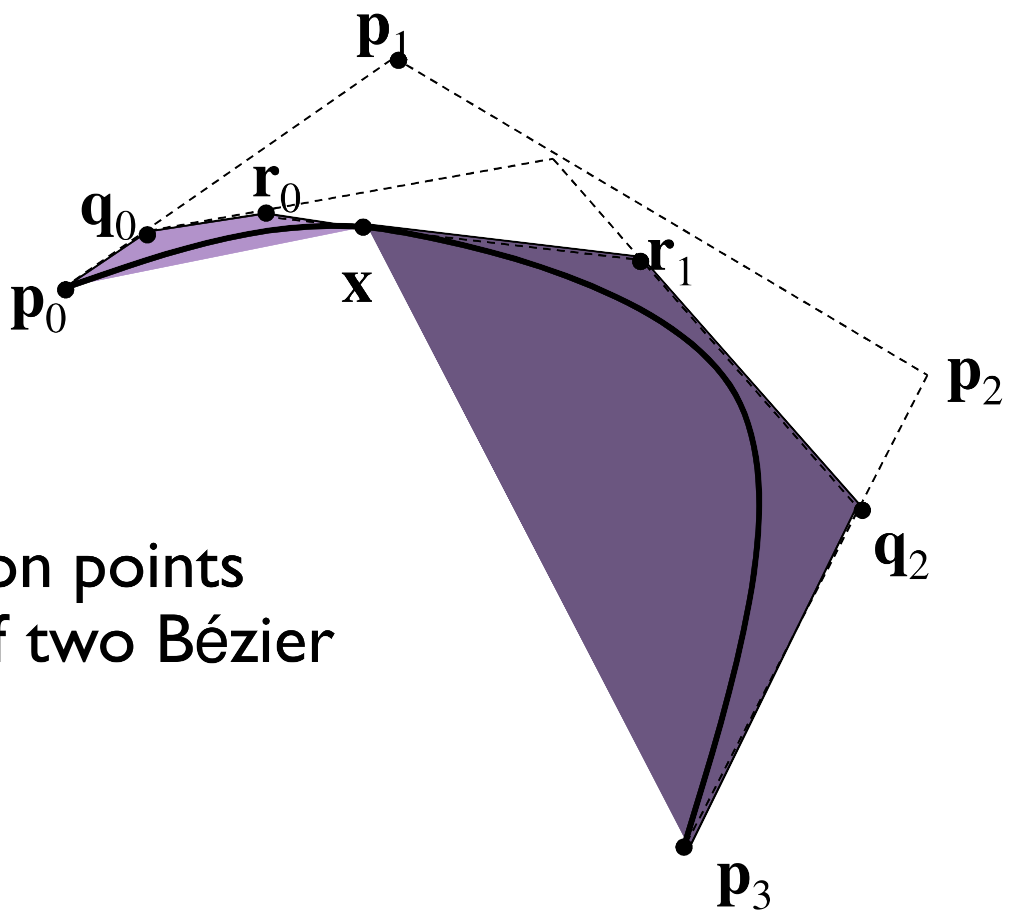
- Use only as many line segments as you need
  - Fewer segments where curve is mostly flat
  - More segments where curve bends
  - Segments never smaller than a pixel



Recursive Subdivision

- Any cubic curve segment can be expressed as a Bézier curve
- Any piece of a cubic curve is itself a cubic curve
- Therefore:
  - Any Bézier curve can be broken down into smaller Bézier curves

De Casteljau Subdivision



- De Casteljau construction points are the control points of two Bézier sub-segments

Adaptive Subdivision Algorithm

- Use De Casteljau construction to split Bézier segment in half
- For each half
  - If "flat enough": draw line segment
  - Else: recurse
- Curve is flat enough if hull is flat enough
  - Test how far the approximating control points are from a straight segment
    - If less than one pixel, the hull is flat enough

Drawing Bézier Curves With OpenGL

- Indirect OpenGL support for drawing curves:
  - Define evaluator map (`glMap`)
  - Draw line strip by evaluating map (`glEvalCoord`)
  - Optimize by pre-computing coordinate grid (`glMapGrid` and `glEvalMesh`)
- More details about OpenGL implementation:
  - http://www.cs.duke.edu/courses/fall09/cps124/notes/12_curves/opengl_nurbs.pdf

Lecture Overview

- Polynomial Curves
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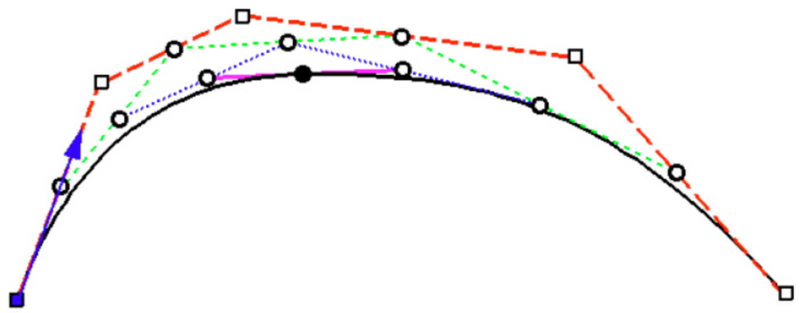
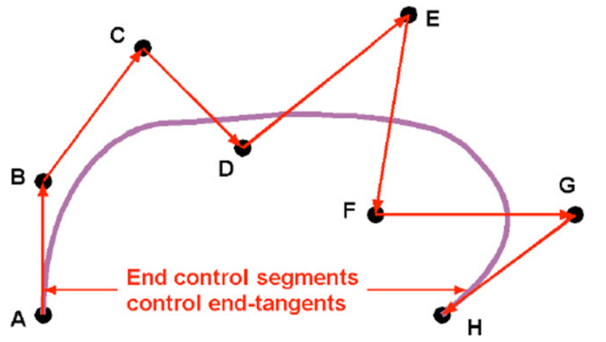
  - Piecewise Bézier curves

More Control Points

- Cubic Bézier curve limited to 4 control points
  - Cubic curve can only have one inflection (point where curve changes direction of bending)
  - Need more control points for more complex curves
- k-1 order Bézier curve with k control points

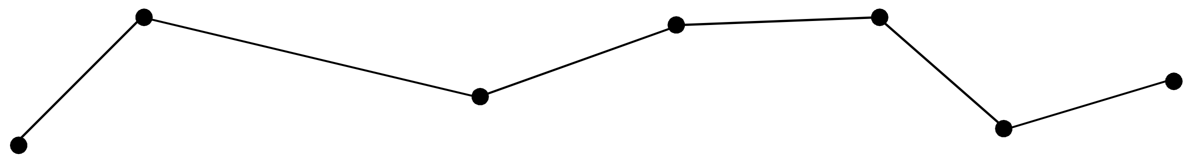




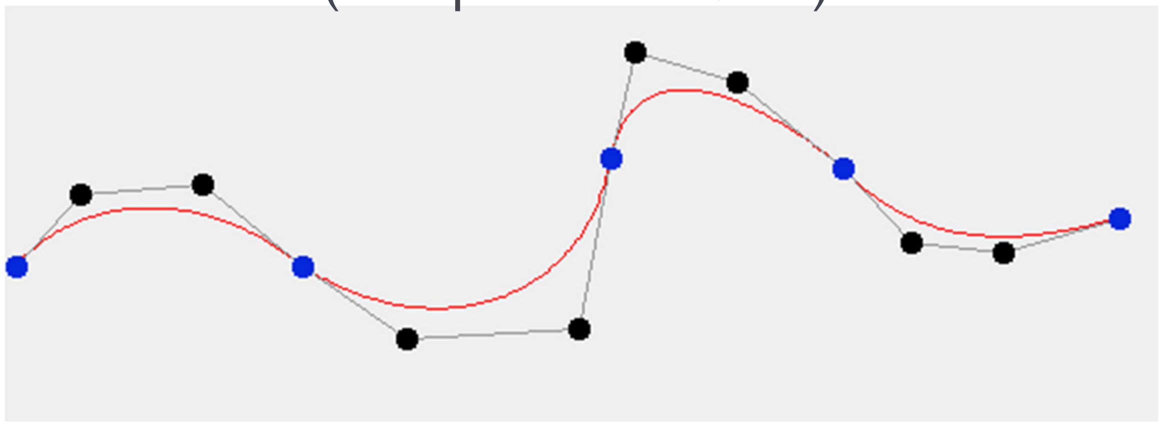
- Hard to control and hard to work with
  - Intermediate points don't have obvious effect on shape
  - Changing any control point changes the whole curve
  - Want *local support*: each control point only influences nearby portion of curve

Piecewise Curves

- Sequence of line segments
  - *Piecewise linear* curve

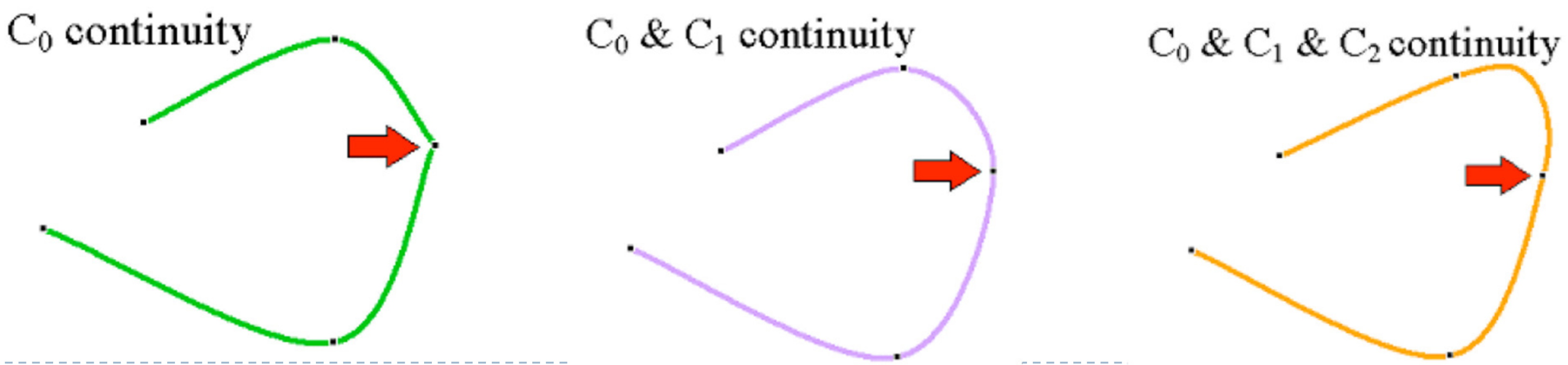


- Sequence of simple (low-order) curves, end-to-end
  - Known as a *piecewise polynomial curve*
- Sequence of cubic curve segments
  - *Piecewise cubic* curve (here piecewise Bézier)



Parametric Continuity

- C^0 continuity:
  - Curve segments are connected
- C^1 continuity:
  - C^0 & 1st-order derivatives agree
  - Curves have same tangents
  - Relevant for smooth shading
- C^2 continuity:
  - C^1 & 2nd-order derivatives agree
  - Curves have same tangents and curvature
  - Relevant for high quality reflections



Geometric Continuity

- G^0:
  - Curve segments are connected
  - Same as C^0
- G^1:
  - G^0 & 1st-order derivatives are proportional at joints
  - Proportional = same direction but may have different magnitudes
  - Weaker than C^1
- G^2:
  - G^1 & 2nd-order derivative proportional at joints

Global Parameterization

- Given N curve segments $\mathbf{x}_0(t)$, $\mathbf{x}_1(t)$, …, $\mathbf{x}_{N-1}(t)$
- Each is parameterized for t from 0 to 1
- Define a piecewise curve
  - Global parameter u from 0 to N

$$\mathbf{x}(u) = \begin{cases} \mathbf{x}_0(u), & 0 \le u \le 1 \\ \mathbf{x}_1(u-1), & 1 \le u \le 2 \\ \vdots & \vdots \\ \mathbf{x}_{N-1}(u-(N-1)), & N-1 \le u \le N \end{cases}$$

$$\mathbf{x}(u) = \mathbf{x}_i(u-i), \text{ where } i = \lfloor u \rfloor \quad (\text{and } \mathbf{x}(N) = \mathbf{x}_{N-1}(1))$$

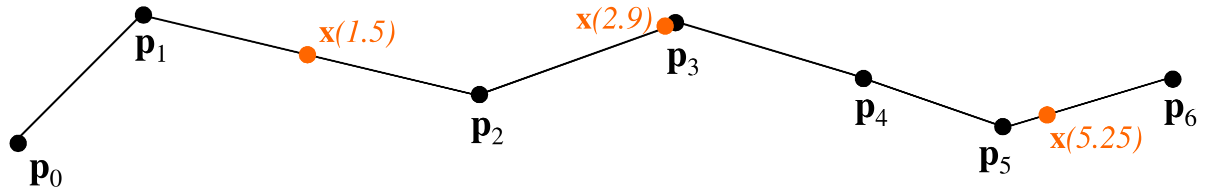
- Alternate: solution u also goes from 0 to 1

$$\mathbf{x}(u) = \mathbf{x}_i(Nu-i), \text{ where } i = \lfloor Nu \rfloor$$

Piecewise-Linear Curve

- Given N+1 points $\mathbf{p}_0, \mathbf{p}_1, \ldots, \mathbf{p}_N$
- Define curve

$$\begin{aligned}\mathbf{x}(u) &= Lerp(u-i, \mathbf{p}_i, \mathbf{p}_{i+1}), \qquad i \le u \le i+1 \\ &= (1-u+i)\mathbf{p}_i + (u-i)\mathbf{p}_{\mathbf{i+1}}, \quad i = \lfloor u \rfloor\end{aligned}$$



- N+1 points define N linear segments
- $\mathbf{x}(i)=\mathbf{p}_i$
- C^0 continuous by construction
- C^1 at $\mathbf{p}_i$ when $\mathbf{p}_i - \mathbf{p}_{i-1} = \mathbf{p}_{i+1} - \mathbf{p}_i$

Piecewise Bézier curve

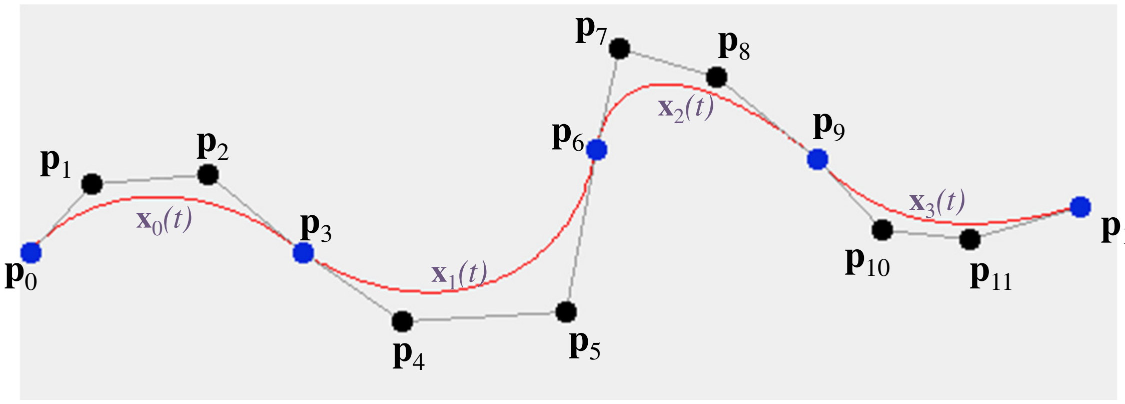
- Given $3N+1$ points $\mathbf{p}_0, \mathbf{p}_1, \ldots, \mathbf{p}_{3N}$
- Define N Bézier segments:

$$\mathbf{x}_0(t) = B_0(t)\mathbf{p}_0 + B_1(t)\mathbf{p}_1 + B_2(t)\mathbf{p}_2 + B_3(t)\mathbf{p}_3$$

$$\mathbf{x}_1(t) = B_0(t)\mathbf{p}_3 + B_1(t)\mathbf{p}_4 + B_2(t)\mathbf{p}_5 + B_3(t)\mathbf{p}_6$$

$$\vdots$$

$$\mathbf{x}_{N-1}(t) = B_0(t)\mathbf{p}_{3N-3} + B_1(t)\mathbf{p}_{3N-2} + B_2(t)\mathbf{p}_{3N-1} + B_3(t)\mathbf{p}_{3N}$$

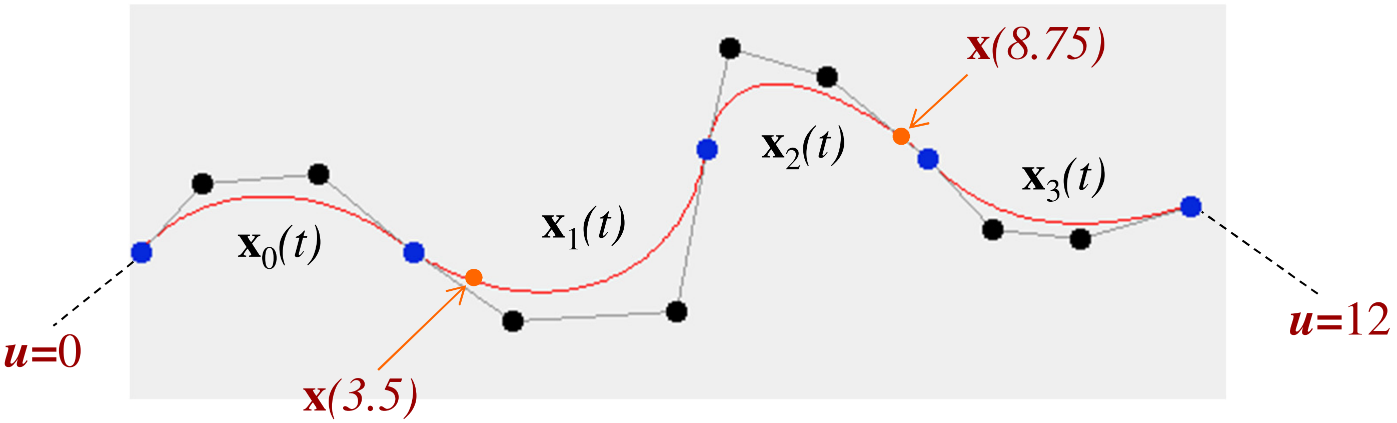


Piecewise Bézier Curve

- Parameter in $0 <= u <= 3N$

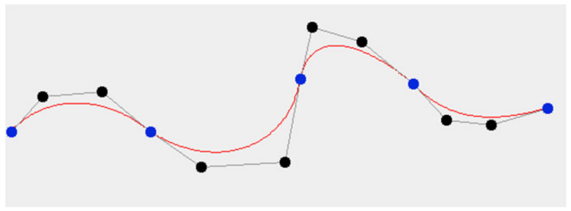
$$\mathbf{x}(u) = \begin{cases} \mathbf{x}_0(\tfrac{1}{3}u), & 0 \le u \le 3 \\ \mathbf{x}_1(\tfrac{1}{3}u - 1), & 3 \le u \le 6 \\ \vdots & \vdots \\ \mathbf{x}_{N-1}(\tfrac{1}{3}u - (N-1)), & 3N-3 \le u \le 3N \end{cases}$$

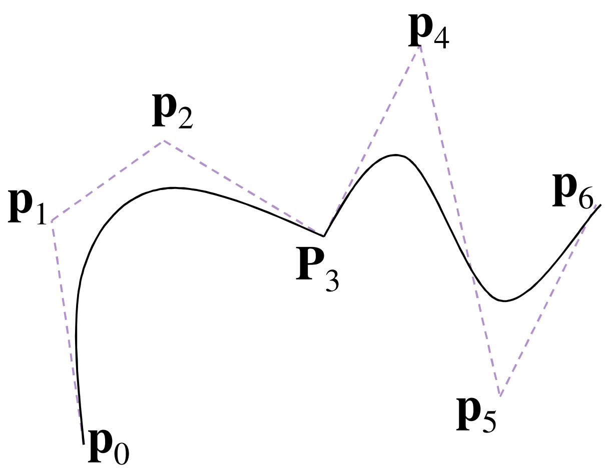
$\mathbf{x}(u) = \mathbf{x}_i\left(\tfrac{1}{3}u - i\right)$, where $i = \left\lfloor \tfrac{1}{3}u \right\rfloor$



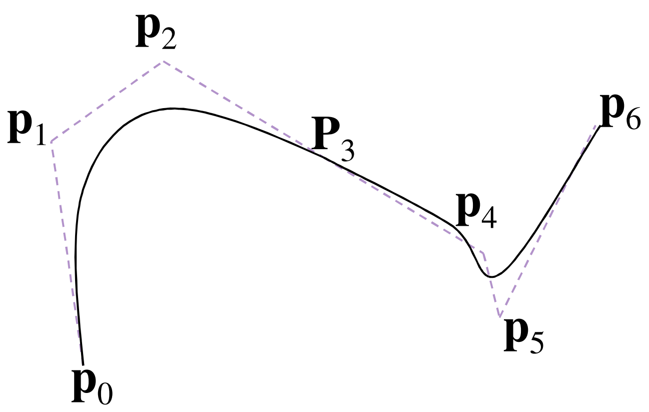
Piecewise Bézier Curve

- $3N+1$ points define N Bézier segments
- $\mathbf{x}(3i)=\mathbf{p}_{3i}$
- C_0 continuous by construction
- C_1 continuous at $\mathbf{p}_{3i}$ when $\mathbf{p}_{3i} - \mathbf{p}_{3i-1} = \mathbf{p}_{3i+1} - \mathbf{p}_{3i}$
- C_2 is harder to achieve





C₁ discontinuous



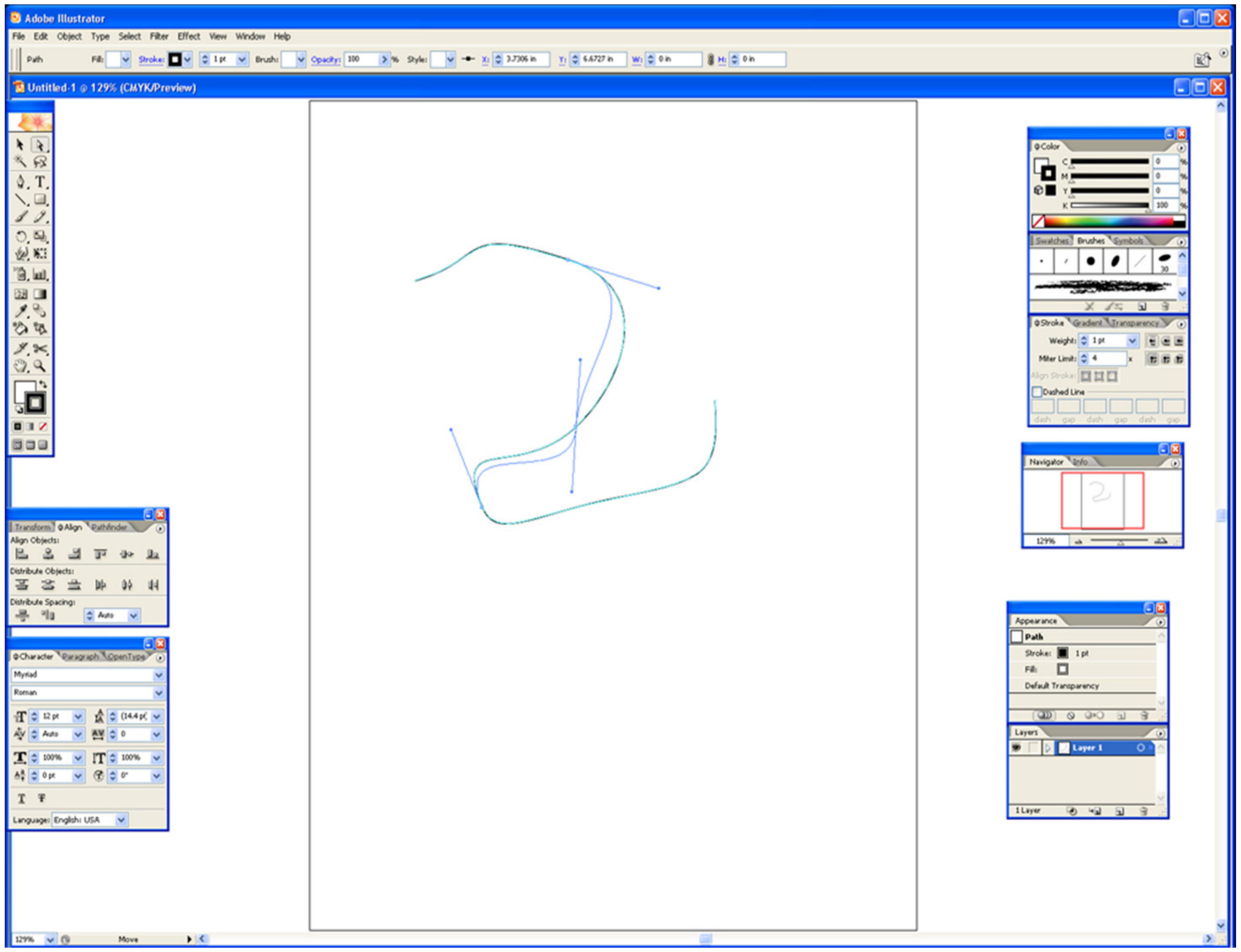
C_1 continuous

Piecewise Bézier Curves

- Used often in 2D drawing programs
- Inconveniences
  - Must have 4 or 7 or 10 or 13 or … (1 plus a multiple of 3) control points
  - Some points interpolate, others approximate
  - Need to impose constraints on control points to obtain C^1 continuity
  - C_2 continuity more difficult
- Solutions
  - User interface using "Bézier handles"
  - Generalization to B-splines or NURBS

Bézier Handles

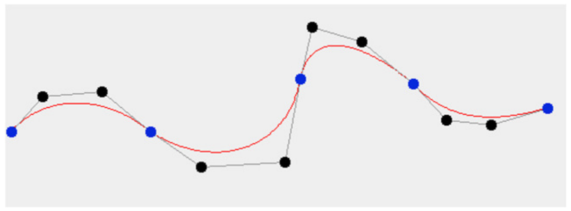
- Segment end points (interpolating) presented as curve control points
- Midpoints (approximating points) presented as "handles"
- Can have option to enforce C_1 continuity

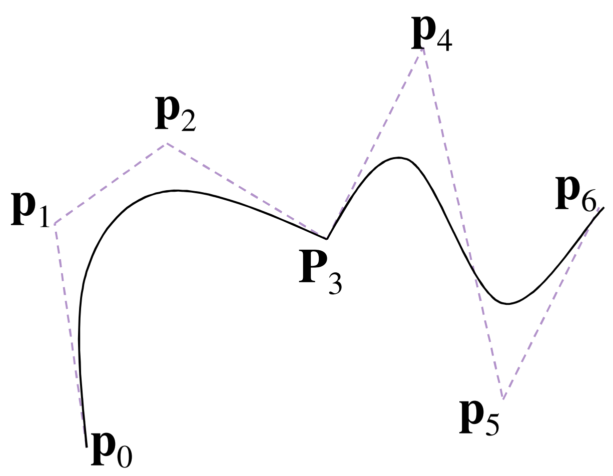

Adobe Illustrator
File Edit Object Type Select Filter Effect View Window Help
Untitled-1 @ 129% (CMYK/Preview)

Adobe Illustrator

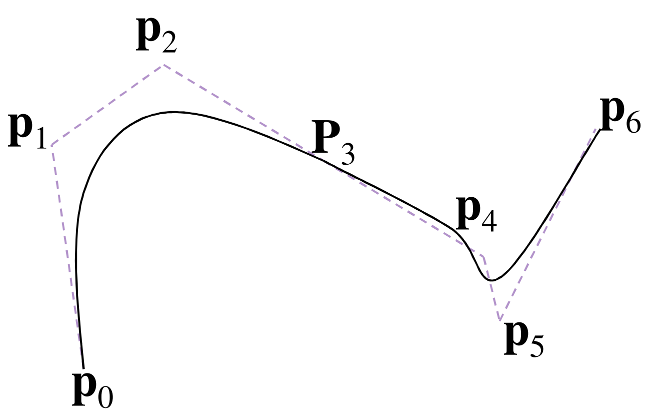
Piecewise Bézier Curve

- $3N+1$ points define N Bézier segments
- $\mathbf{x}(3i)=\mathbf{p}_{3i}$
- C_0 continuous by construction
- C_1 continuous at $\mathbf{p}_{3i}$ when $\mathbf{p}_{3i} - \mathbf{p}_{3i-1} = \mathbf{p}_{3i+1} - \mathbf{p}_{3i}$
- C_2 is harder to achieve





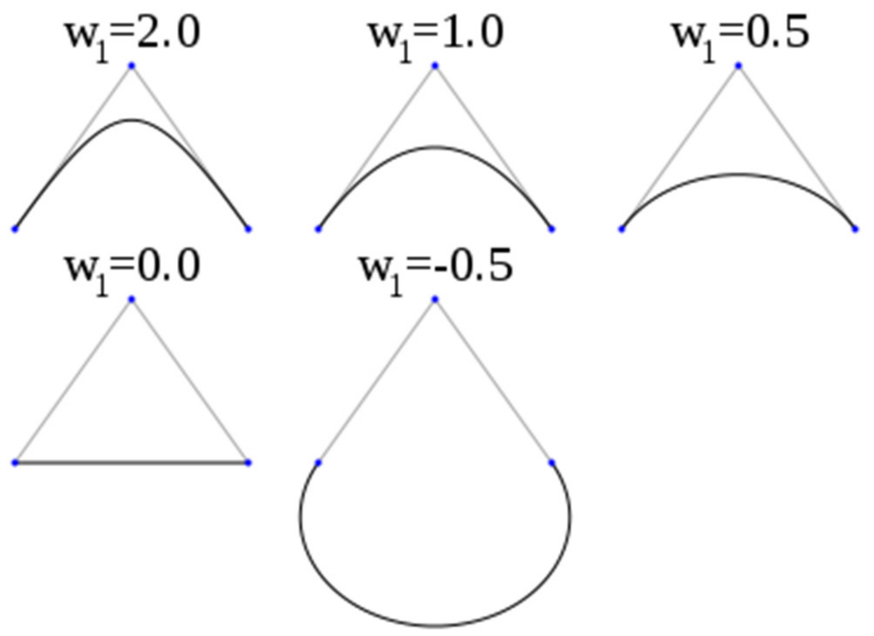
C_1 discontinuous



C_1 continuous

Rational Curves

- Weight causes point to “pull” more (or less)
- Can model circles with proper points and weights,
- Below: rational quadratic Bézier curve (three control points)



B-Splines

- B as in **B**asis-Splines
- Basis is blending function
- Difference to Bézier blending function:
  - B-spline blending function can be zero outside a particular range (limits scope over which a control point has influence)
- B-Spline is defined by control points and range in which each control point is active.

NURBS

- **N**on **U**niform **R**ational **B-S**plines
- Generalization of Bézier curves
- Non uniform:
- Combine B-Splines (limited scope of control points) and Rational Curves (weighted control points)
- Can exactly model conic sections (circles, ellipses)
- OpenGL support: see `gluNurbsCurve`
- http://bentonian.com/teaching/AdvGraph0809/demos/Nurbs2 /index.html
- http://mathworld.wolfram.com/NURBSCurve.html