CSE 167:
Introduction to Computer Graphics
Lecture #12: Bezier Curves

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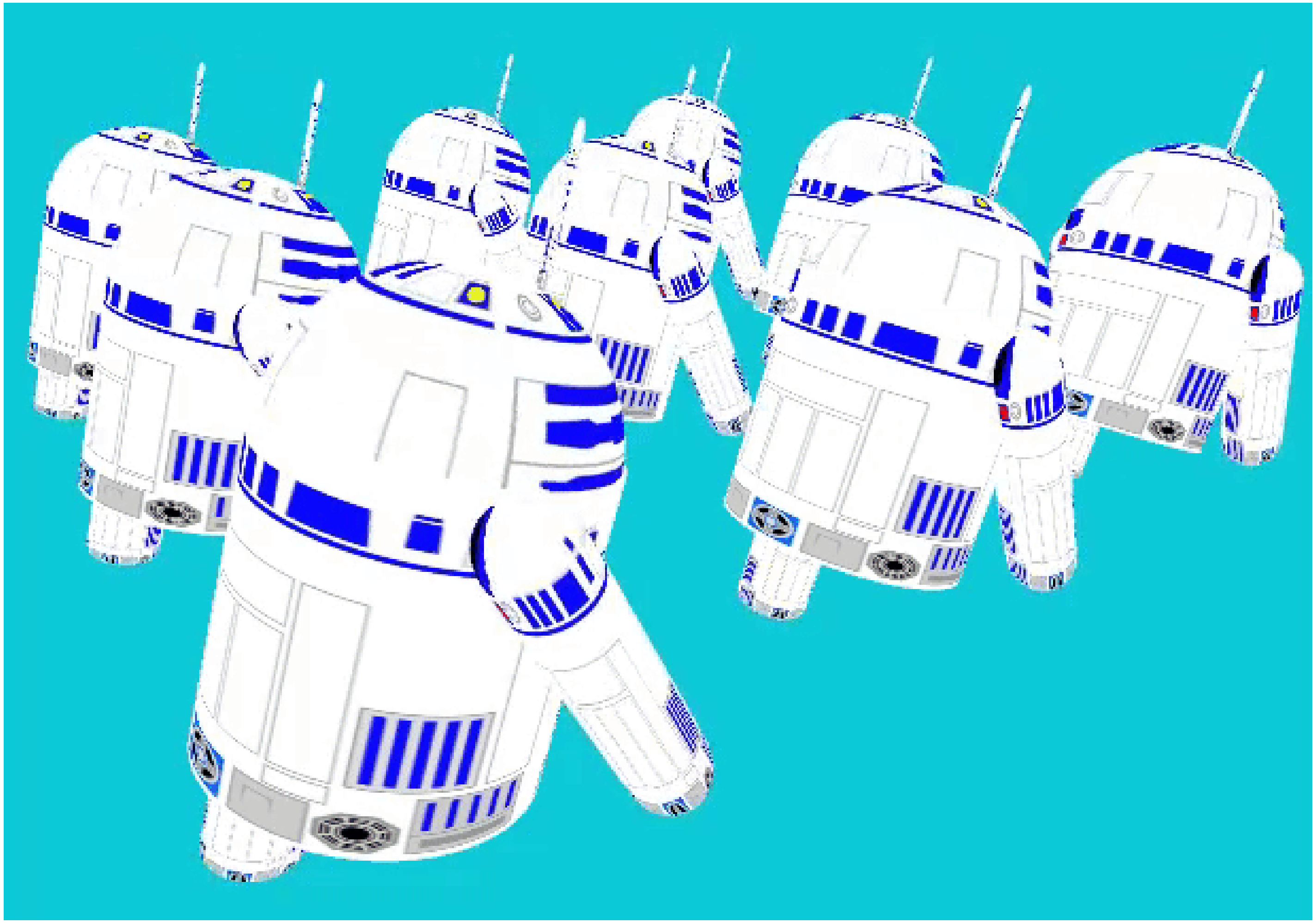
Fall Quarter 2018

Announcements

- Project 3 late grading this Friday
- Project 4 due next Friday
  - Grading in CSE basement labs B260 and B270
  - Upload code to TritonEd by 2pm
  - Grading order managed by Autograder
- Best robot competition results

4^{th} Place: Unnamed by Gregory Sabado



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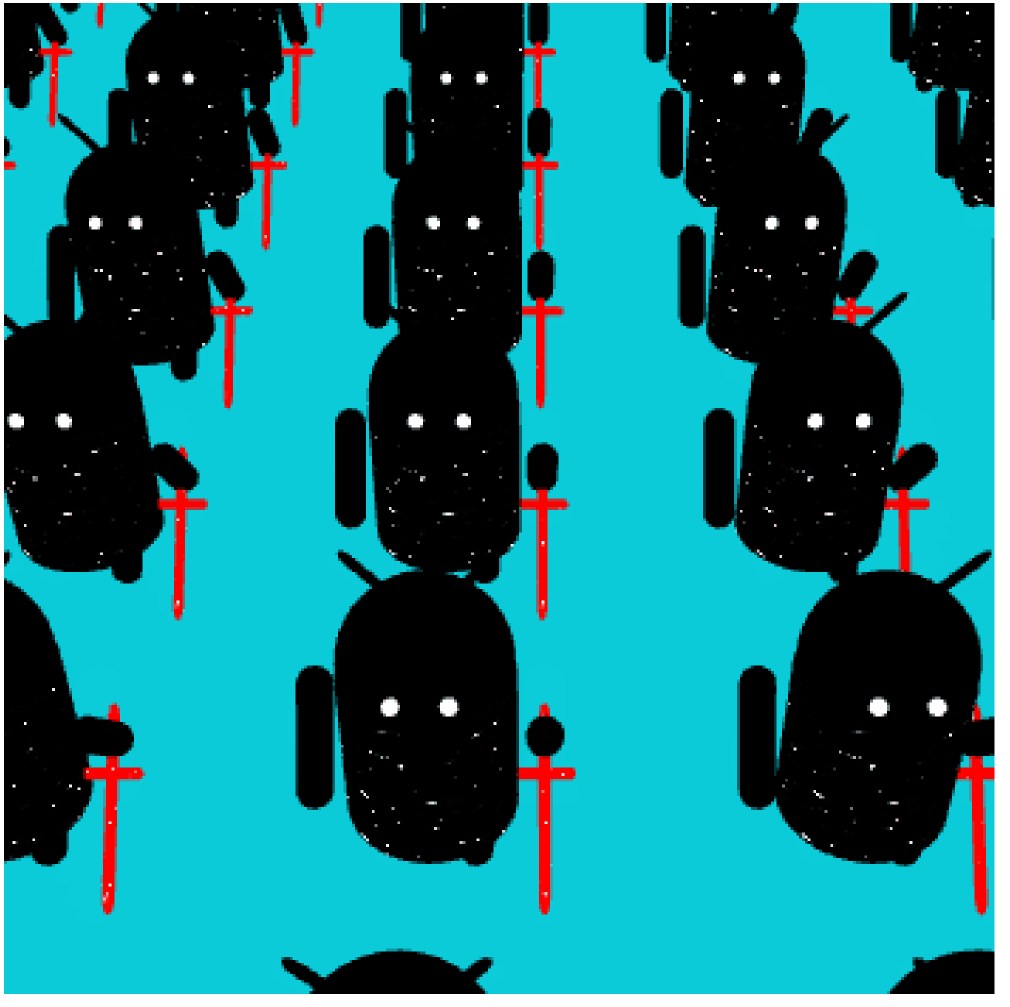
4th Place: "'No one will stand in our way' - Kylo Ren" by Richard Du



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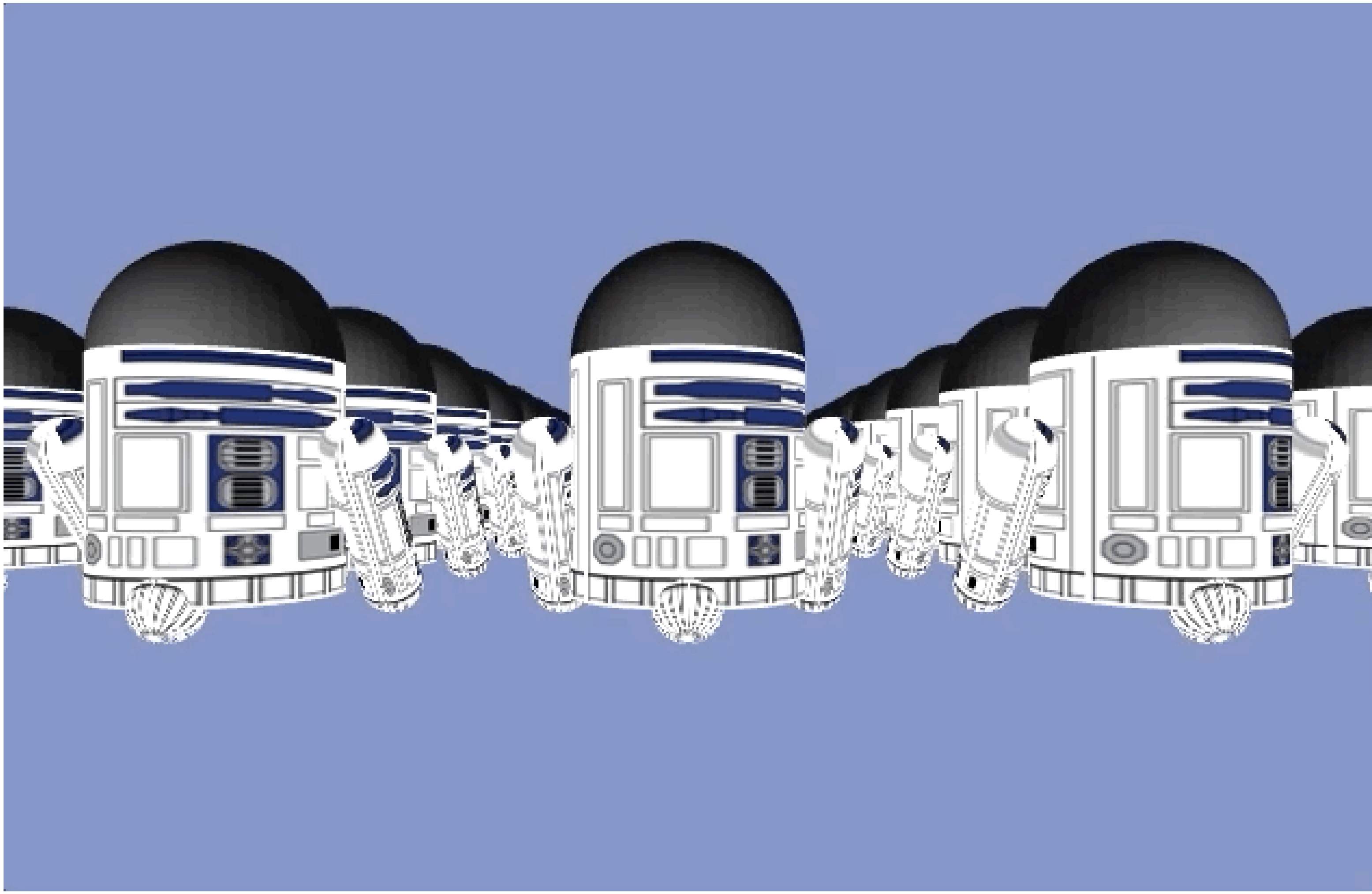
4th Place: "R2D2" by Eliana Schulner



3rd Place:
"Wall-E" by Pedro Sousa Meireles

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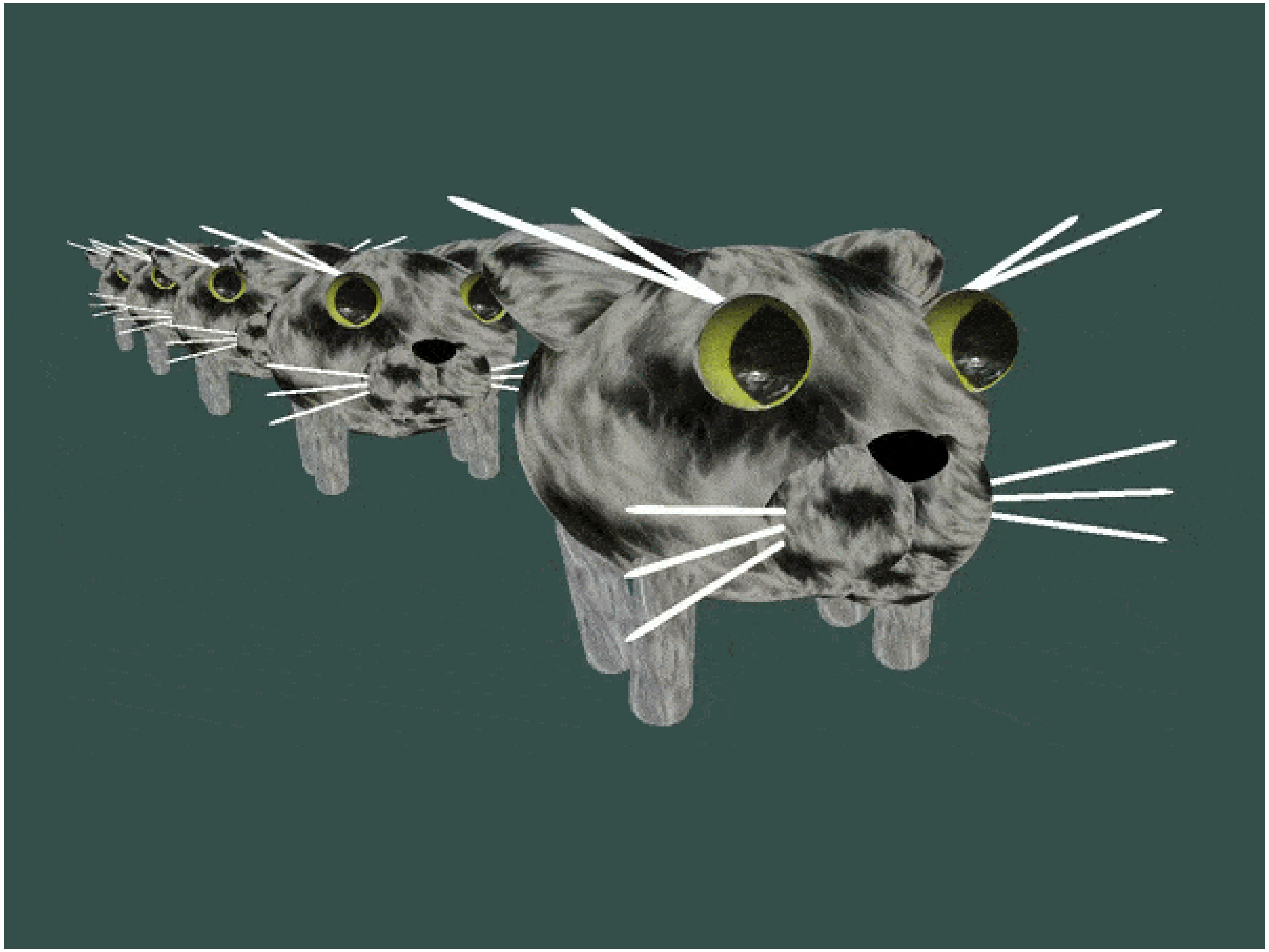
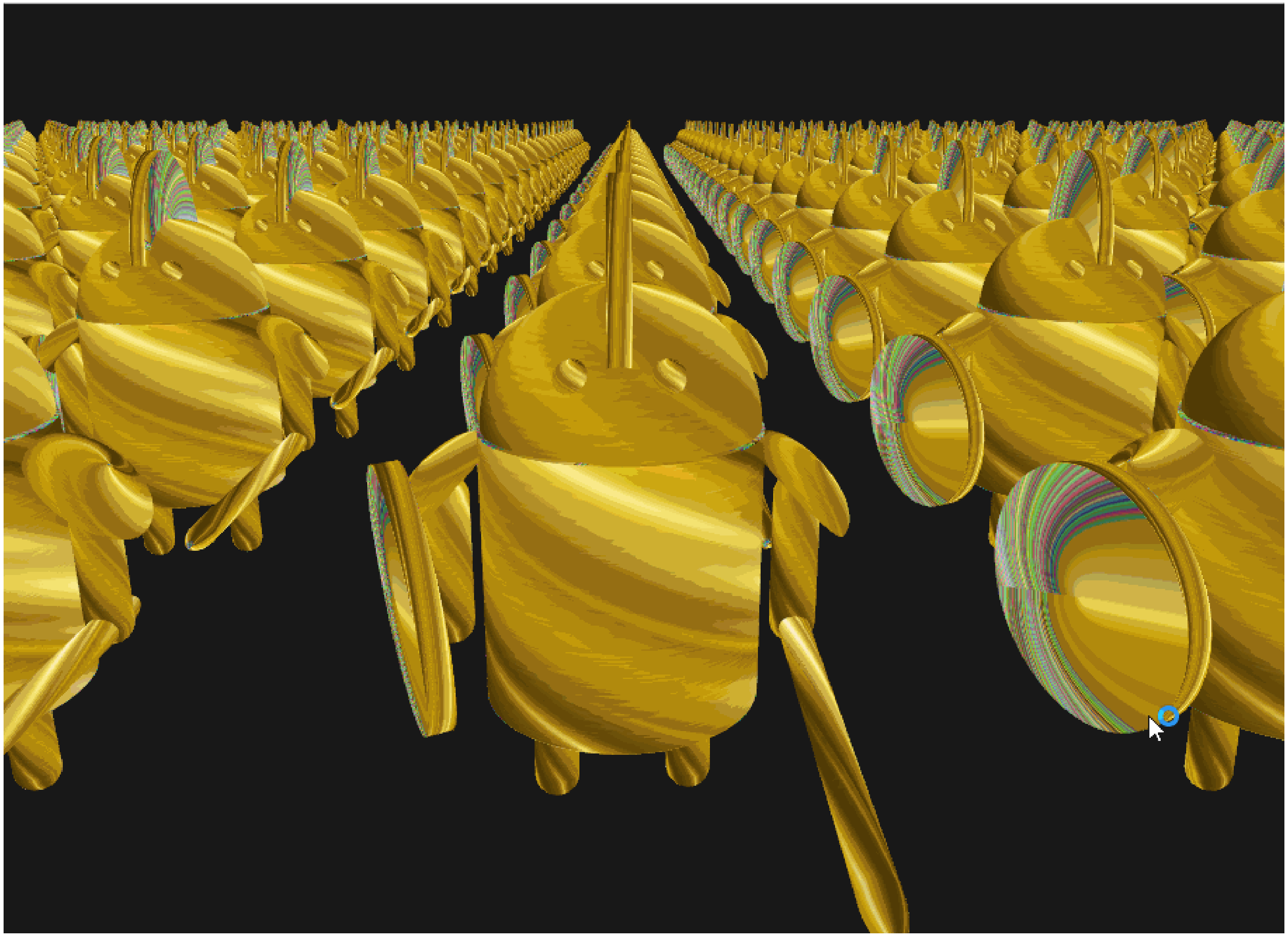
2nd Place: "meow" by Conor



1st Place: "This is Sparta!" by Haoqi Wu

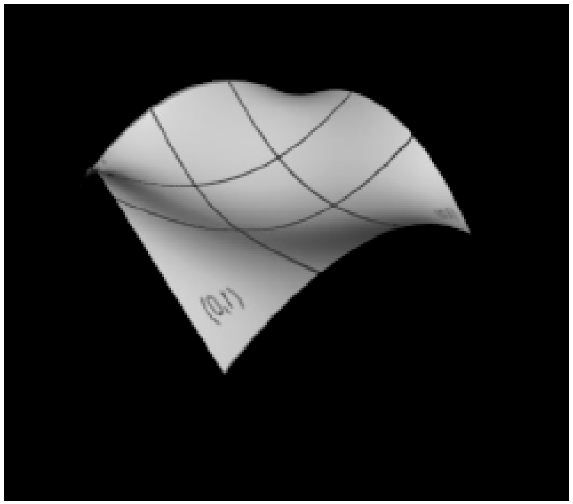
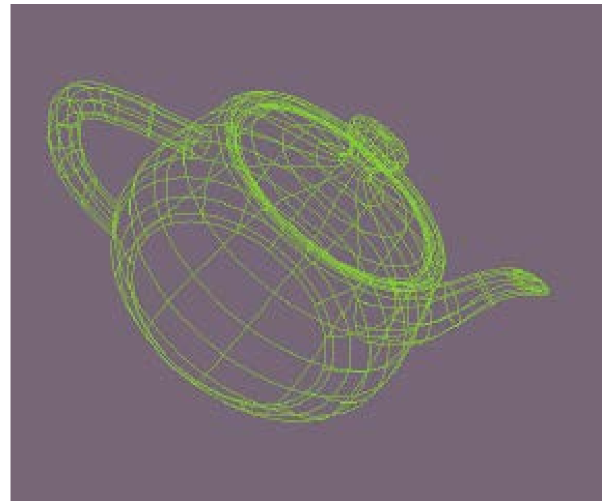


Lecture Overview

- Polynomial Curves
  - Introduction
  - Polynomial functions
- Bézier Curves
  - Introduction
  - Drawing Bézier curves
  - Piecewise Bézier curves

Curves

- Can be generalized to surface patches

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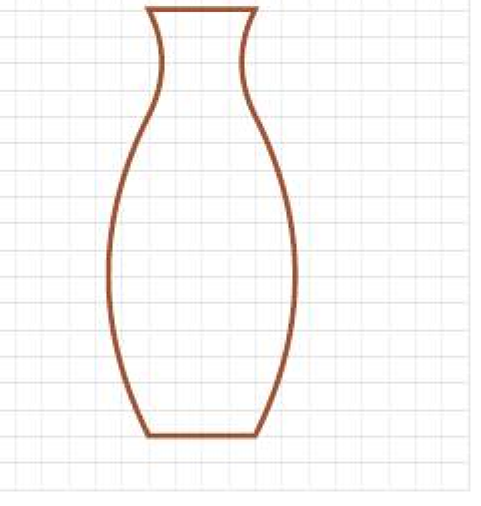
Curve Representation

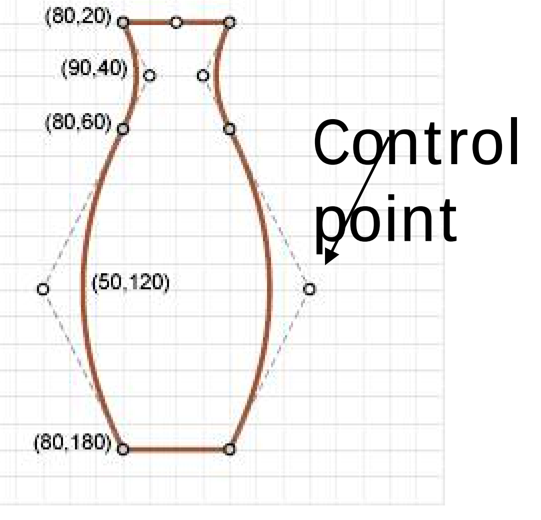
Why not specify many points along a curve and connect with lines:

- Can't get smooth results when magnified – more points needed
- Large storage and CPU requirements

Instead: specify a curve with a small number of "control points"

- Known as a *spline curve* or *spline.*





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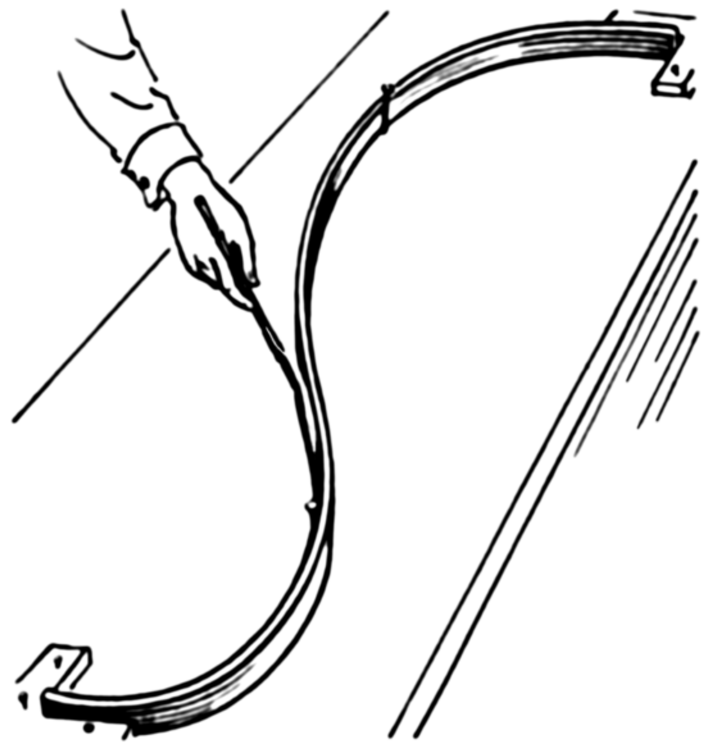
Spline: Definition

- Wikipedia:
  - Term comes from flexible spline devices used by shipbuilders and draftsmen to draw smooth shapes.
  - Spline consists of a long strip fixed in position at a number of points that relaxes to form a smooth curve passing through those points.



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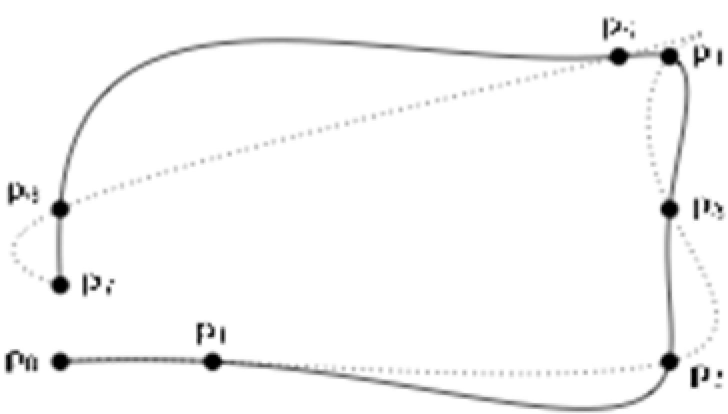
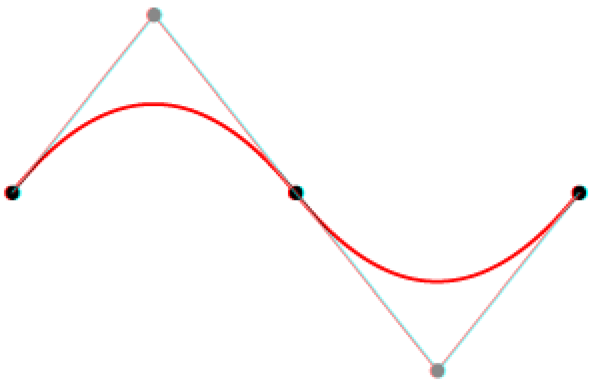
Interpolating Control Points

- "Interpolating" means that curve goes through all control points
- A.k.a. "Anchor Points"
- Seems most intuitive
- But hard to control exact behavior



Approximating Control Points

- Curve is "influenced" by control points



- Various types
- Most common: polynomial functions
  - Bézier spline (our focus)
  - B-spline (generalization of Bézier spline)
  - NURBS (Non Uniform Rational Basis Spline): used in CAD tools

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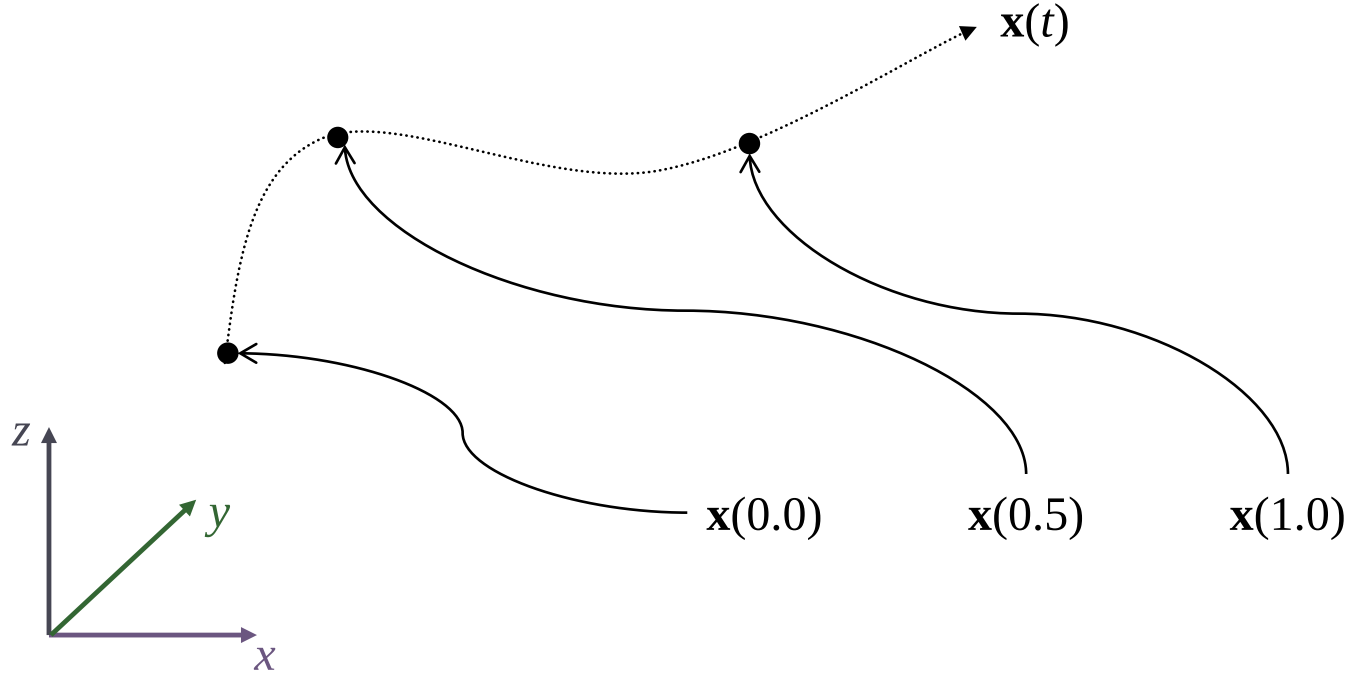
Mathematical Definition

- A vector valued function of one variable $\mathbf{x}(t)$
  - Given t, compute a 3D point $\mathbf{x}=(x,y,z)$
  - Could be interpreted as three functions: $x(t)$, $y(t)$, $z(t)$
  - Parameter t "moves a point along the curve"



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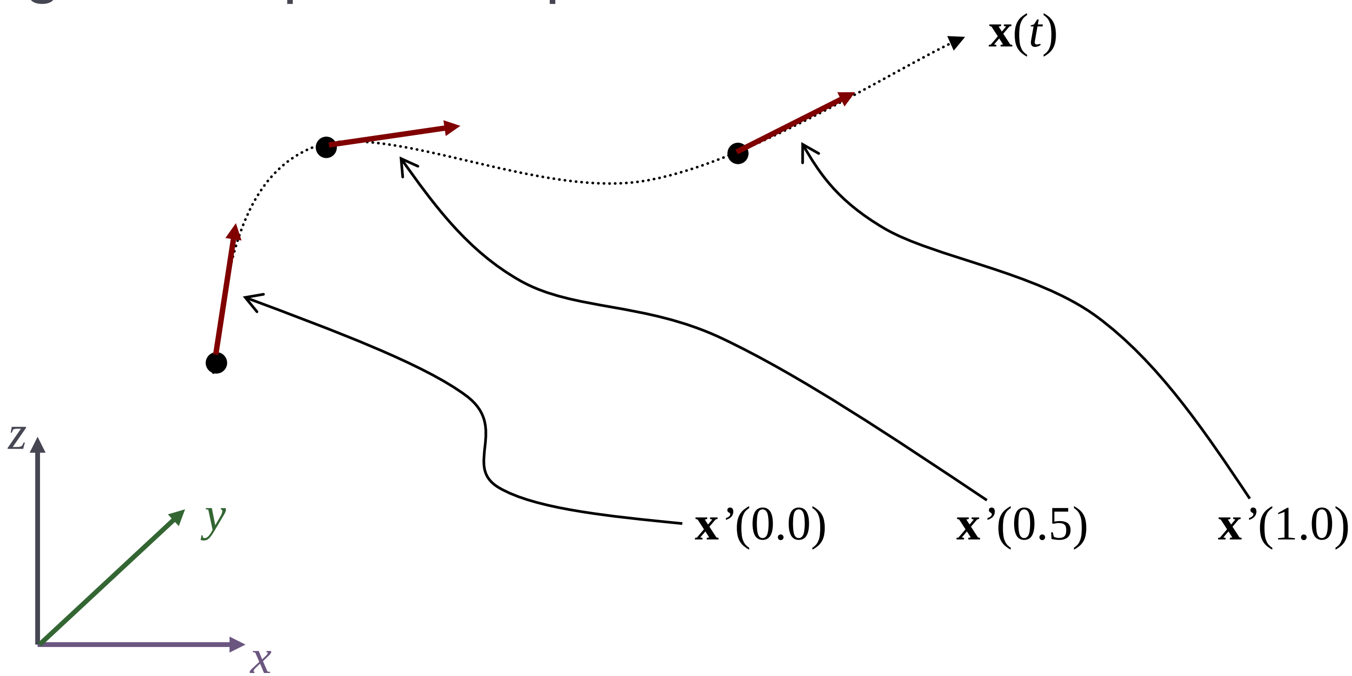
Tangent Vector

- Derivative $\mathbf{x}'(t) = \dfrac{d\mathbf{x}}{dt} = (x'(t), y'(t), z'(t))$
- Vector x':
  - Points in direction of movement
  - Length corresponds to speed



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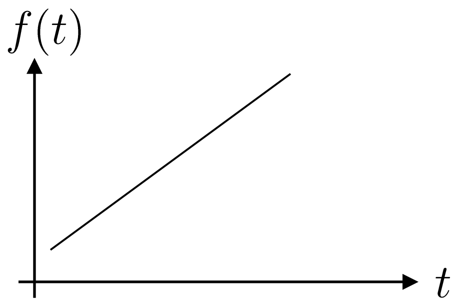
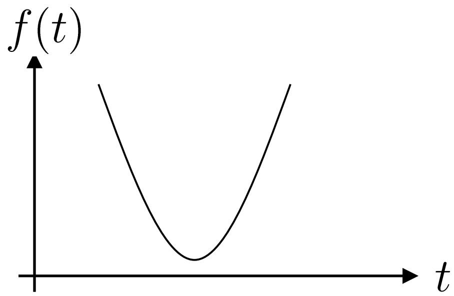
Polynomial Functions

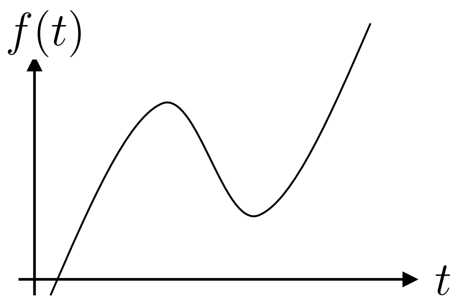
- Linear: (1st order) $$f(t) = at + b$$



- Quadratic: (2nd order) $$f(t) = at^2 + bt + c$$



- Cubic: (3rd order) $$f(t) = at^3 + bt^2 + ct + d$$



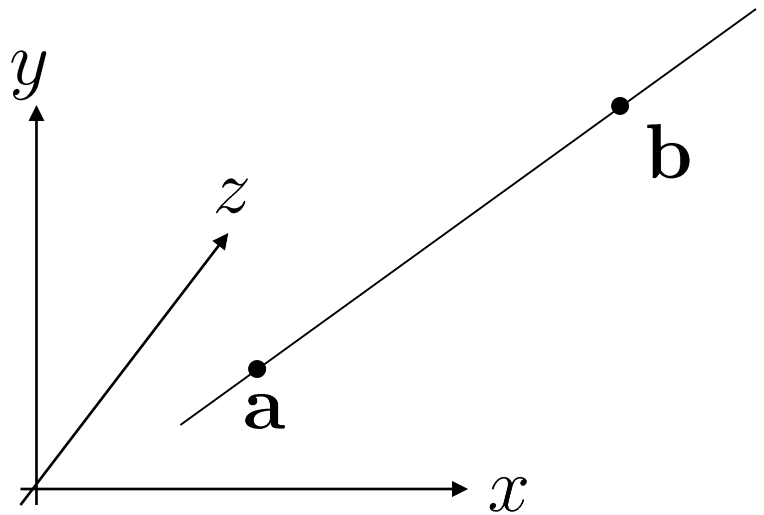
Polynomial Curves in 3D

- Linear $\mathbf{x}(t) = \mathbf{a}t + \mathbf{b}$

$$\mathbf{x} = (x, y, z), \mathbf{a} = (a_x, a_y, a_z), \mathbf{b} = (b_x, b_y, b_z)$$

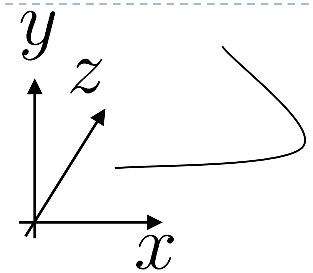
- Evaluated as:

$$x(t) = a_x t + b_x$$
$$y(t) = a_y t + b_y$$
$$z(t) = a_z t + b_z$$

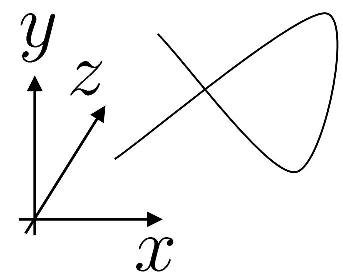


Polynomial Curves in 3D

- Quadratic: $\mathbf{x}(t) = \mathbf{a}t^2 + \mathbf{b}t + \mathbf{c}$
  (2nd order)



- Cubic: $\mathbf{x}(t) = \mathbf{a}t^3 + \mathbf{b}t^2 + \mathbf{c}t + \mathbf{d}$
  (3rd order)



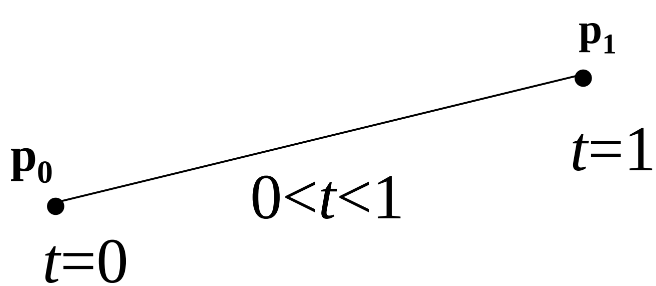
- We usually define the curve for $0 \leq t \leq 1$

Control Points

- Polynomial coefficients **a, b, c, d** can be interpreted as *control points*
  - Remember: **a, b, c, d** have x,y,z components each
- But: they do not intuitively describe the shape of the curve
- Goal: intuitive control points

Weighted Average

- Based on linear interpolation (LERP)
  - Weighted average between two values
  - "Value" could be a number, vector, color, …
- Interpolate between points $\mathbf{p}_0$ and $\mathbf{p}_1$ with parameter t
  - Defines a "curve" that is straight (first-order spline)

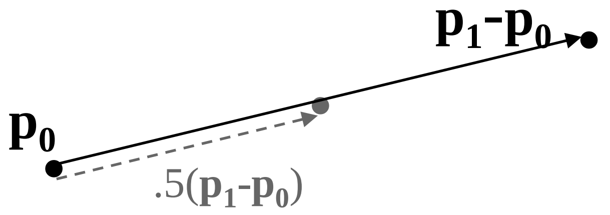


$$\mathbf{x}(t) = Lerp\left(t,\ \mathbf{p}_0,\ \mathbf{p}_1\right) = \left(1 - t\right)\mathbf{p}_0 + t\ \mathbf{p}_1$$

Linear Polynomial

$$\mathbf{x}(t) = \underbrace{(\mathbf{p}_1 - \mathbf{p}_0)}_{\substack{\text{vector} \\ \mathbf{a}}}\, t + \underbrace{\mathbf{p}_0}_{\substack{\text{point} \\ \mathbf{b}}}$$

- Curve is based at point $\mathbf{p_0}$
- Add the vector, scaled by t



Matrix Form

$$\mathbf{x}(t) = \begin{bmatrix} \mathbf{p}_0 & \mathbf{p}_1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} t \\ 1 \end{bmatrix} = \mathbf{GBT}$$

- Geometry matrix $\mathbf{G} = \begin{bmatrix} \mathbf{p}_0 & \mathbf{p}_1 \end{bmatrix}$
- Geometric basis $\mathbf{B} = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$
- Polynomial basis $T = \begin{bmatrix} t \\ 1 \end{bmatrix}$
- In components

$$\mathbf{x}(t) = \begin{bmatrix} p_{0x} & p_{1x} \\ p_{0y} & p_{1y} \\ p_{0z} & p_{1z} \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} t \\ 1 \end{bmatrix}$$

Summary

1. Grouped by points $\mathbf{p}$: weighted average

$$\mathbf{x}(t) = \mathbf{p}_0(1-t) + \mathbf{p}_1 t$$

2. Grouped by t: linear polynomial

$$\mathbf{x}(t) = (\mathbf{p}_1 - \mathbf{p}_0)t + \mathbf{p}_0$$

3. Matrix form:

$$\mathbf{x}(t) = \begin{bmatrix} \mathbf{p}_0 & \mathbf{p}_1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} t \\ 1 \end{bmatrix}$$

Tangent

- Weighted average $\mathbf{x}'(t) = (-1)\mathbf{p}_0 + (+1)\mathbf{p}_1$
- Polynomial $\mathbf{x}'(t) = 0t + (\mathbf{p}_1 - \mathbf{p}_0)$
- Matrix form $\mathbf{x}'(t) = \begin{bmatrix} \mathbf{p}_0 & \mathbf{p}_1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

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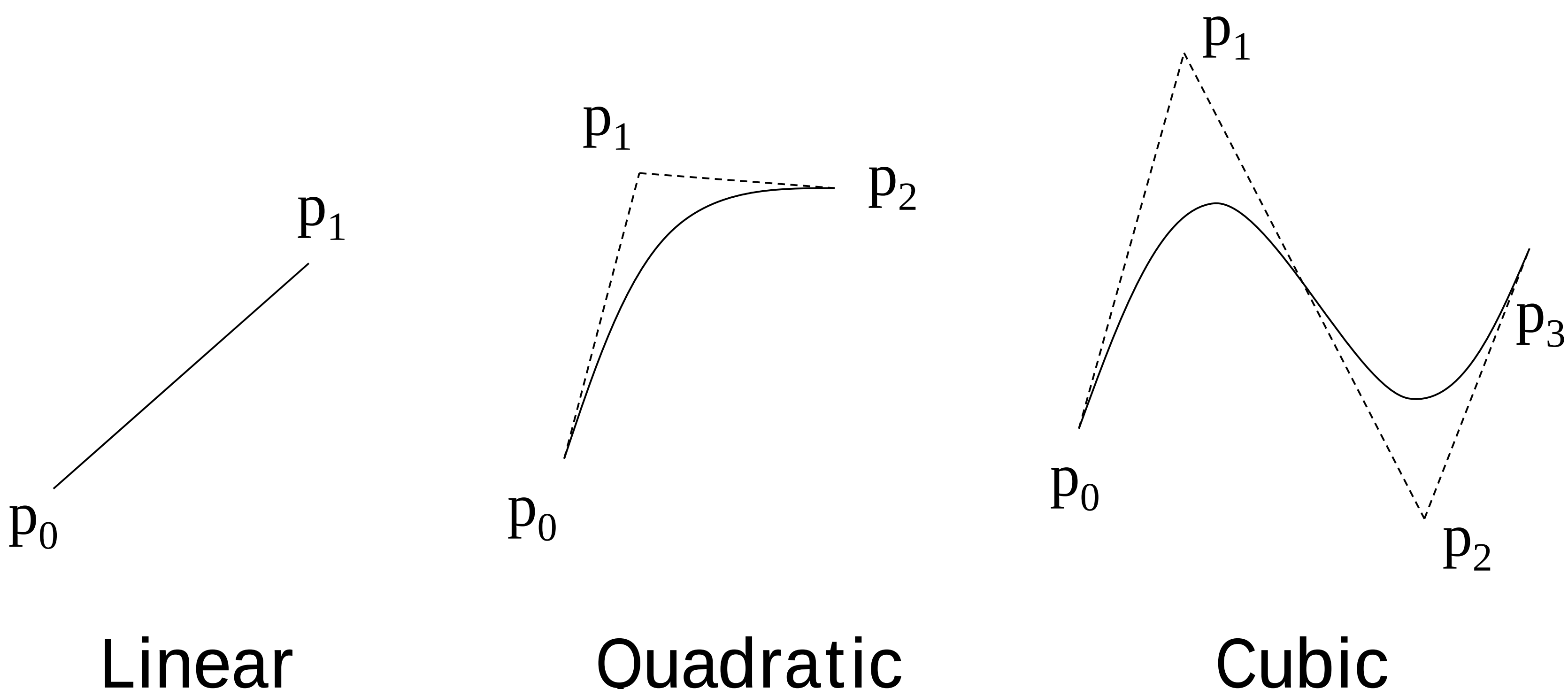
Bézier Curves

- Are a higher order extension of linear interpolation



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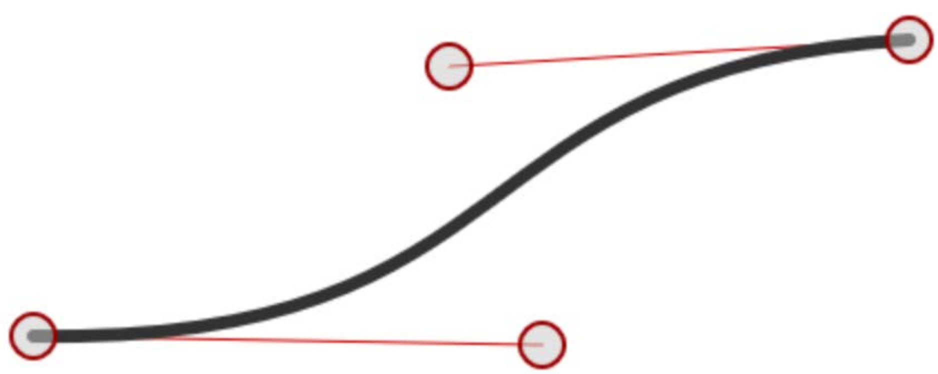
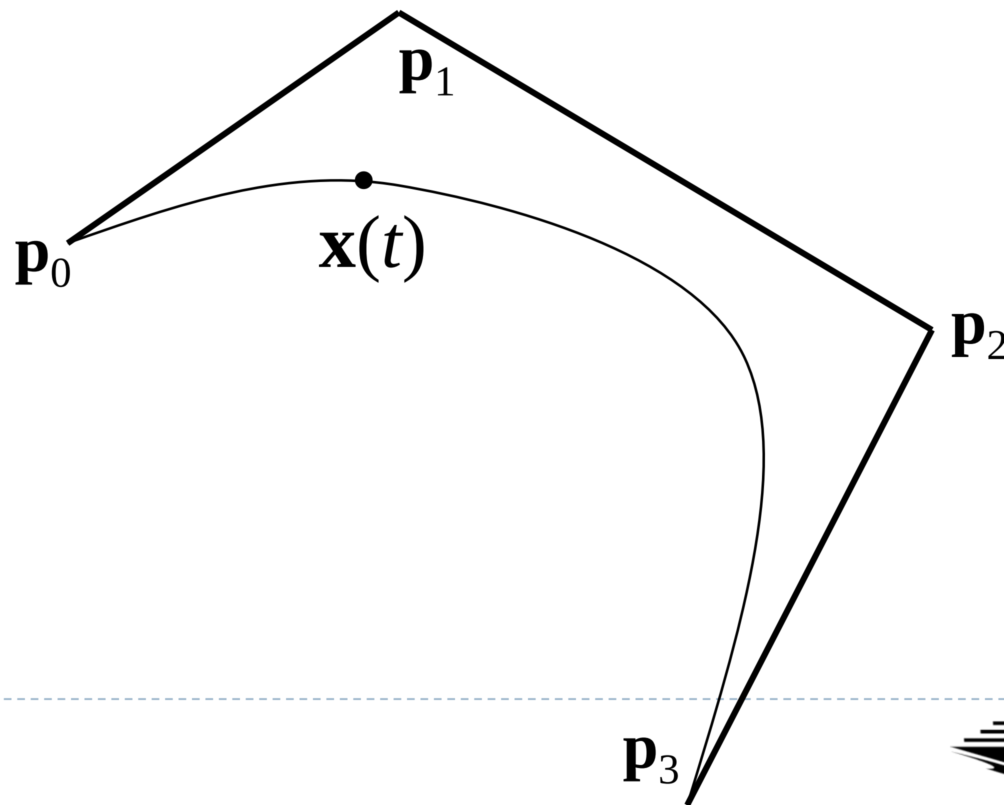
Bézier Curves

- Give intuitive control over curve with control points
  - Endpoints are interpolated, intermediate points are approximated
- Demo:
  - http://blogs.sitepointstatic.com/examples/tech/canvas-curves/bezier-curve.html



Cubic Bézier Curve

- Most commonly used case
- Defined by four control points:
  - Two interpolated endpoints (points are on the curve)
  - Two points control the tangents at the endpoints
- Points $\mathbf{x}$ on curve defined as function of parameter t



Algorithmic Construction

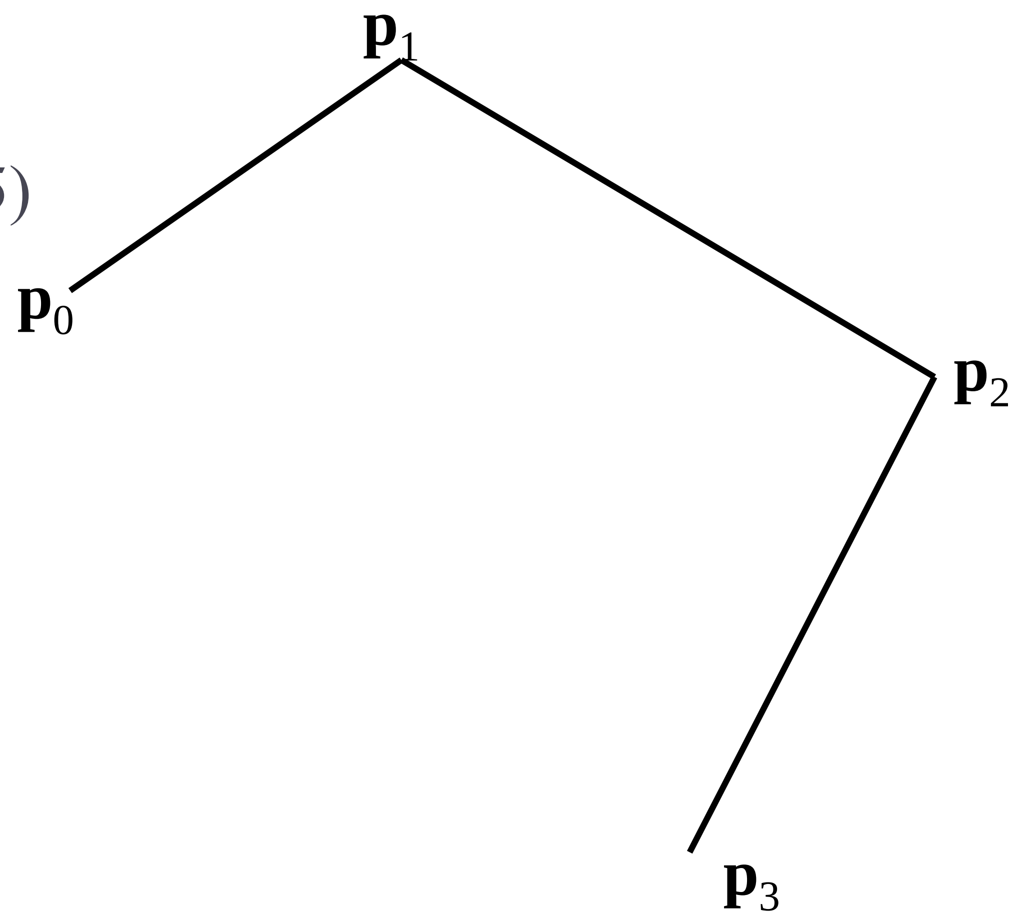
- Algorithmic construction
  - *De Casteljau* algorithm, developed at Citroen in 1959, named after its inventor Paul de Casteljau (pronounced "Cast-all-'Joe")
  - Developed independently from Bézier's work:
    Bézier created the formulation using blending functions, Casteljau devised the recursive interpolation algorithm

De Casteljau Algorithm

- A recursive series of linear interpolations
  - Works for any order Bezier function, not only cubic
- Not very efficient to evaluate
  - Other forms more commonly used
- But:
  - Gives intuition about the geometry
  - Useful for subdivision

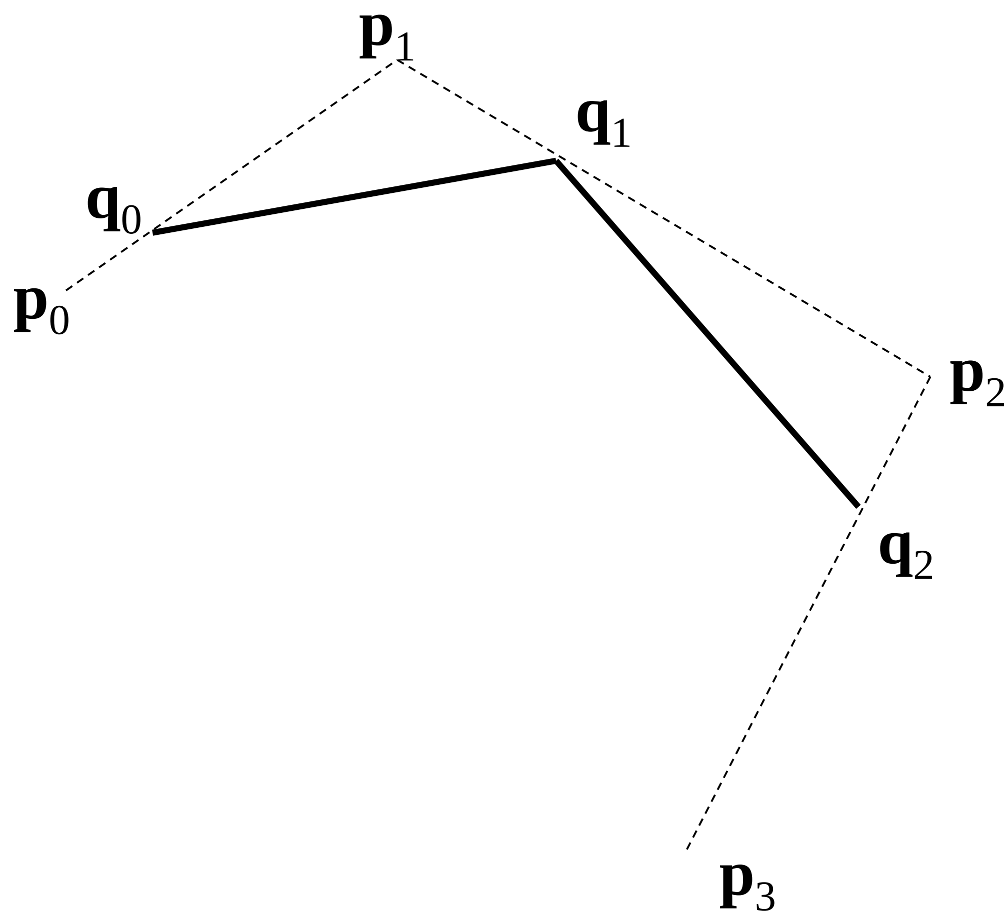
De Casteljau Algorithm

- Given:
  - Four control points
  - A value of t (here $t \approx 0.25$)



De Casteljau Algorithm

$$\mathbf{q}_0(t) = Lerp\left(t, \mathbf{p}_0, \mathbf{p}_1\right)$$
$$\mathbf{q}_1(t) = Lerp\left(t, \mathbf{p}_1, \mathbf{p}_2\right)$$
$$\mathbf{q}_2(t) = Lerp\left(t, \mathbf{p}_2, \mathbf{p}_3\right)$$



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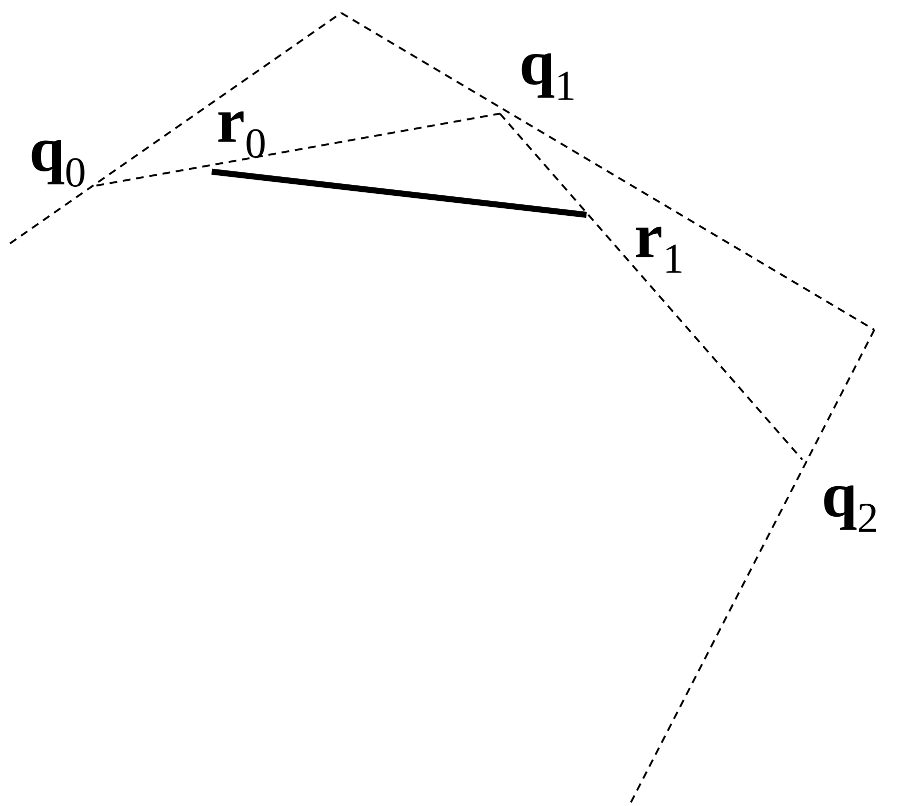
De Casteljau Algorithm

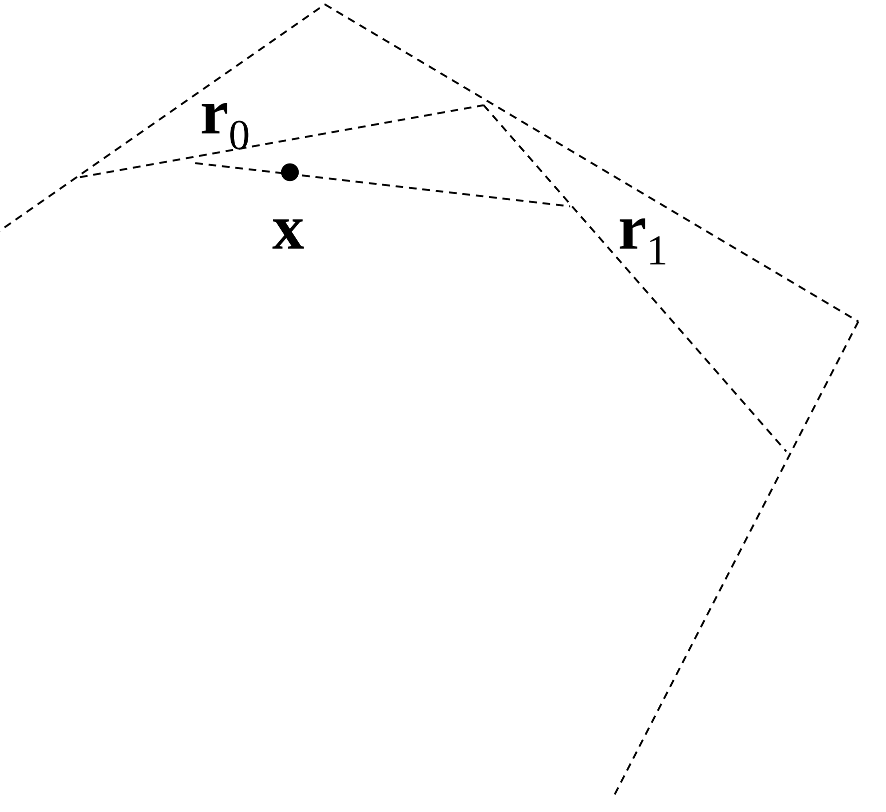


$$\mathbf{r}_0(t) = Lerp\left(t, \mathbf{q}_0(t), \mathbf{q}_1(t)\right)$$

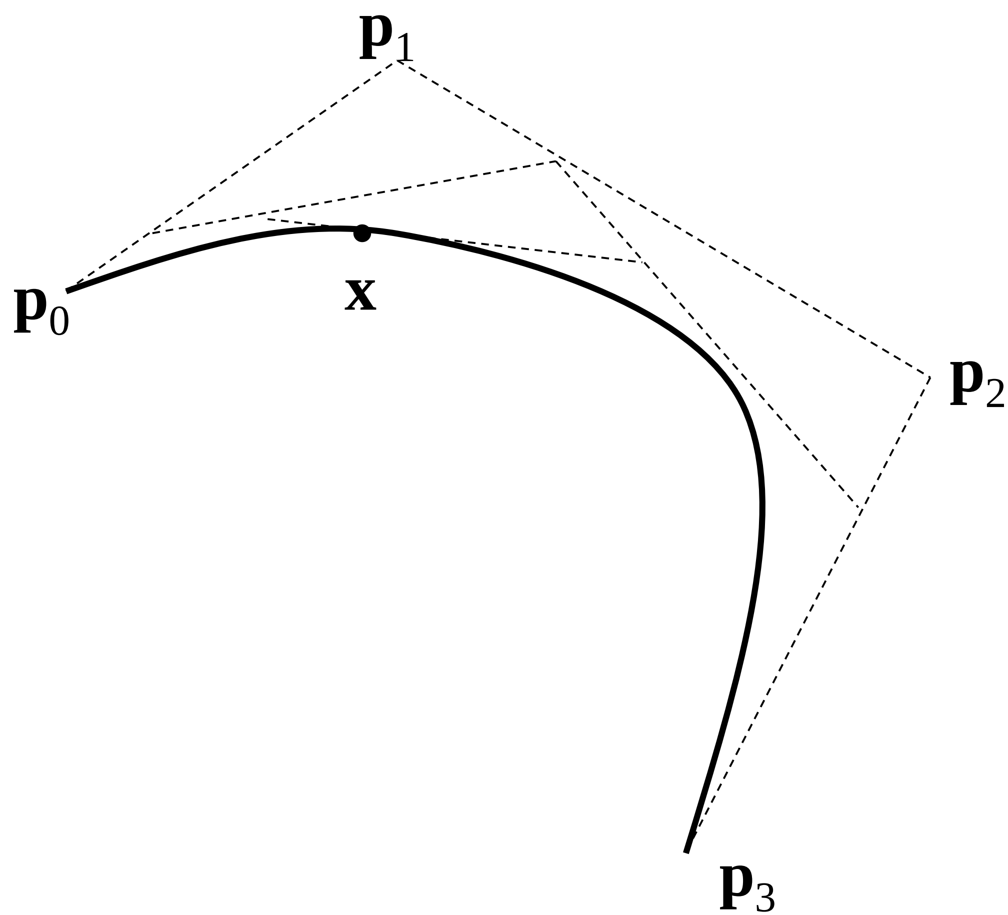
$$\mathbf{r}_1(t) = Lerp\left(t, \mathbf{q}_1(t), \mathbf{q}_2(t)\right)$$

De Casteljau Algorithm

$$\mathbf{x}(t) = Lerp\big(t, \mathbf{r}_0(t), \mathbf{r}_1(t)\big)$$



De Casteljau Algorithm



- Demo
  - https://www.jasondavies.com/animated-bezier/

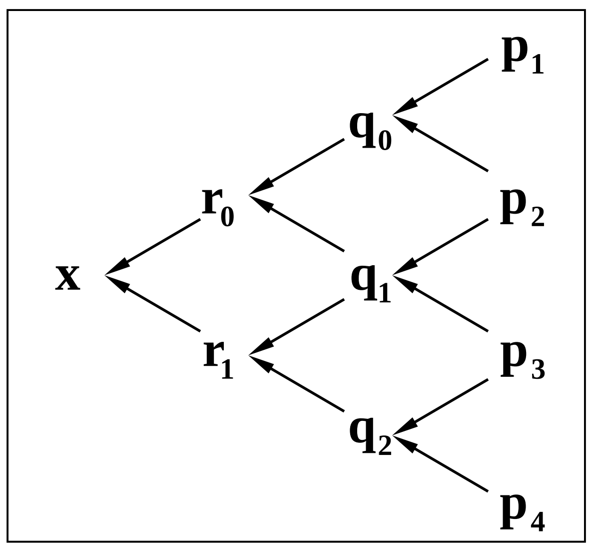
Recursive Linear Interpolation

$$\mathbf{x} = Lerp\left(t, \mathbf{r}_0, \mathbf{r}_1\right)$$

$$\mathbf{r}_0 = Lerp\left(t, \mathbf{q}_0, \mathbf{q}_1\right)$$
$$\mathbf{r}_1 = Lerp\left(t, \mathbf{q}_1, \mathbf{q}_2\right)$$

$$\mathbf{q}_0 = Lerp\left(t, \mathbf{p}_0, \mathbf{p}_1\right)$$
$$\mathbf{q}_1 = Lerp\left(t, \mathbf{p}_1, \mathbf{p}_2\right)$$
$$\mathbf{q}_2 = Lerp\left(t, \mathbf{p}_2, \mathbf{p}_3\right)$$

$$\mathbf{p}_0$$
$$\mathbf{p}_1$$
$$\mathbf{p}_2$$
$$\mathbf{p}_3$$



Expand the LERPs

$$\mathbf{q}_0(t) = Lerp\left(t, \mathbf{p}_0, \mathbf{p}_1\right) = (1-t)\mathbf{p}_0 + t\mathbf{p}_1$$

$$\mathbf{q}_1(t) = Lerp\left(t, \mathbf{p}_1, \mathbf{p}_2\right) = (1-t)\mathbf{p}_1 + t\mathbf{p}_2$$

$$\mathbf{q}_2(t) = Lerp\left(t, \mathbf{p}_2, \mathbf{p}_3\right) = (1-t)\mathbf{p}_2 + t\mathbf{p}_3$$

$$\mathbf{r}_0(t) = Lerp\left(t, \mathbf{q}_0(t), \mathbf{q}_1(t)\right) = (1-t)\left((1-t)\mathbf{p}_0 + t\mathbf{p}_1\right) + t\left((1-t)\mathbf{p}_1 + t\mathbf{p}_2\right)$$

$$\mathbf{r}_1(t) = Lerp\left(t, \mathbf{q}_1(t), \mathbf{q}_2(t)\right) = (1-t)\left((1-t)\mathbf{p}_1 + t\mathbf{p}_2\right) + t\left((1-t)\mathbf{p}_2 + t\mathbf{p}_3\right)$$

$$\begin{aligned}\mathbf{x}(t) &= Lerp\left(t, \mathbf{r}_0(t), \mathbf{r}_1(t)\right)\\ &= (1-t)\left((1-t)\left((1-t)\mathbf{p}_0 + t\mathbf{p}_1\right) + t\left((1-t)\mathbf{p}_1 + t\mathbf{p}_2\right)\right)\\ &\quad + t\left((1-t)\left((1-t)\mathbf{p}_1 + t\mathbf{p}_2\right) + t\left((1-t)\mathbf{p}_2 + t\mathbf{p}_3\right)\right)\end{aligned}$$

Weighted Average of Control Points

- Regroup for p:

$$\mathbf{x}(t) = (1-t)\Big((1-t)\big((1-t)\mathbf{p}_0 + t\mathbf{p}_1\big) + t\big((1-t)\mathbf{p}_1 + t\mathbf{p}_2\big)\Big)$$
$$+t\Big((1-t)\big((1-t)\mathbf{p}_1 + t\mathbf{p}_2\big) + t\big((1-t)\mathbf{p}_2 + t\mathbf{p}_3\big)\Big)$$

$$\mathbf{x}(t) = (1-t)^3\mathbf{p}_0 + 3(1-t)^2 t\mathbf{p}_1 + 3(1-t)t^2\mathbf{p}_2 + t^3\mathbf{p}_3$$

$$\mathbf{x}(t) = \overbrace{\left(-t^3 + 3t^2 - 3t + 1\right)}^{B_0(t)}\mathbf{p}_0 + \overbrace{\left(3t^3 - 6t^2 + 3t\right)}^{B_1(t)}\mathbf{p}_1$$
$$+\underbrace{\left(-3t^3 + 3t^2\right)}_{B_2(t)}\mathbf{p}_2 + \underbrace{\left(t^3\right)}_{B_3(t)}\mathbf{p}_3$$

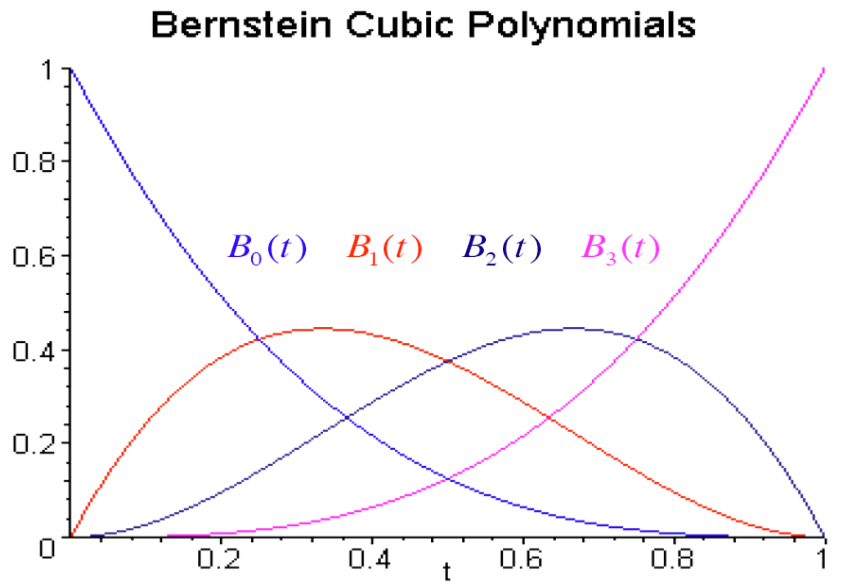
Cubic Bernstein Polynomials

$$\mathbf{x}(t) = B_0(t)\mathbf{p}_0 + B_1(t)\mathbf{p}_1 + B_2(t)\mathbf{p}_2 + B_3(t)\mathbf{p}_3$$

The cubic *Bernstein polynomials :*

$$B_0(t) = -t^3 + 3t^2 - 3t + 1$$
$$B_1(t) = 3t^3 - 6t^2 + 3t$$
$$B_2(t) = -3t^3 + 3t^2$$
$$B_3(t) = t^3$$

$$\sum B_i(t) = 1$$



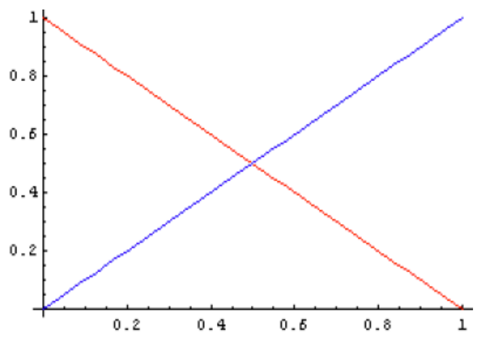
- Weights $B_i(t)$ add up to 1 for any value of t

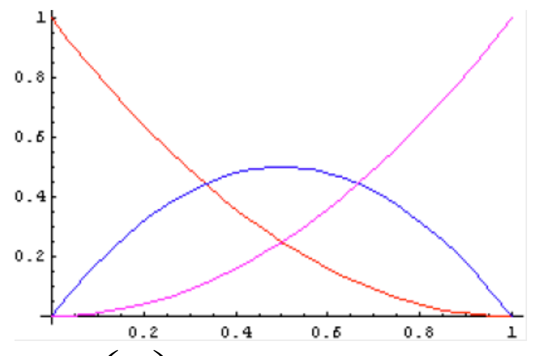
General Bernstein Polynomials

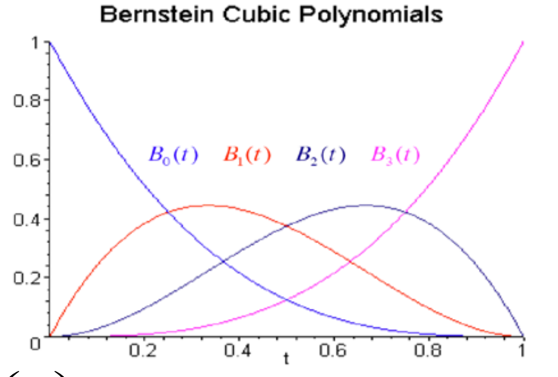
$$B_0^1(t) = -t + 1$$
$$B_1^1(t) = t$$

$$B_0^2(t) = t^2 - 2t + 1$$
$$B_1^2(t) = -2t^2 + 2t$$
$$B_2^2(t) = t^2$$

$$B_0^3(t) = -t^3 + 3t^2 - 3t + 1$$
$$B_1^3(t) = 3t^3 - 6t^2 + 3t$$
$$B_2^3(t) = -3t^3 + 3t^2$$
$$B_3^3(t) = t^3$$







$$B_i^n(t) = \binom{n}{i}(1-t)^{n-i}(t)^i \qquad \binom{n}{i} = \frac{n!}{i!(n-i)!}$$

$$\sum B_i^n(t) = 1$$

n! = factorial of n
(n+1)! = n! x (n+1)

Any order Bézier Curves

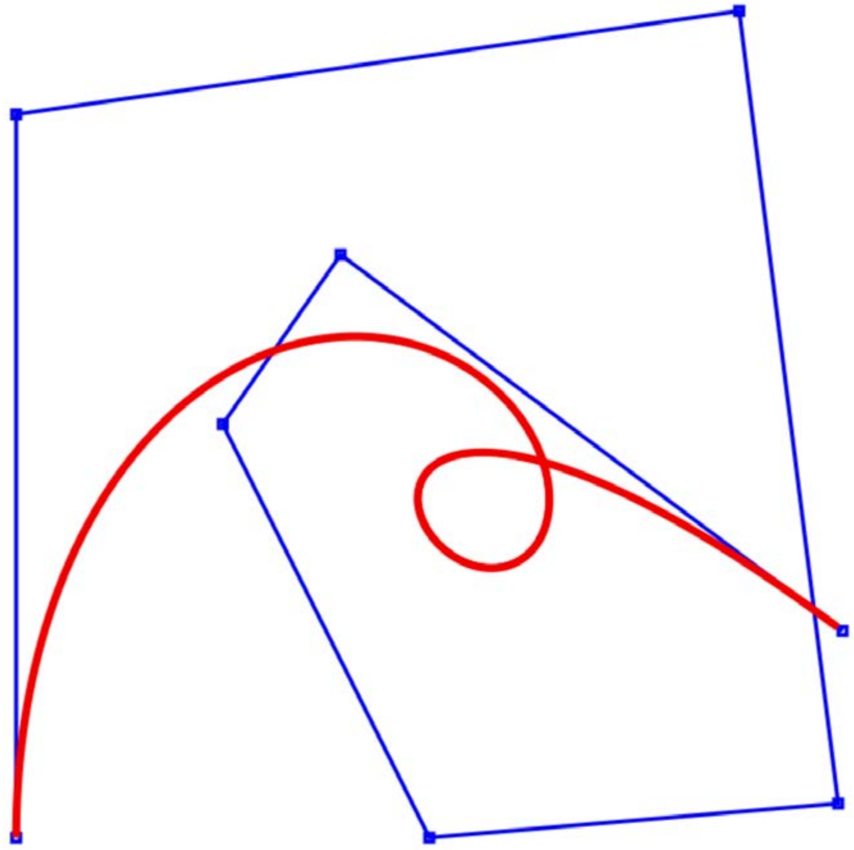
- *n*th-order Bernstein polynomials form *n*th-order Bézier curves

$$B_i^n(t) = \binom{n}{i}(1-t)^{n-i}(t)^i$$

$$\mathbf{x}(t) = \sum_{i=0}^{n} B_i^n(t)\mathbf{p}_i$$

Demo: Bezier curves of multiple orders

- http://www.ibiblio.org/e-notes/Splines/bezier.html

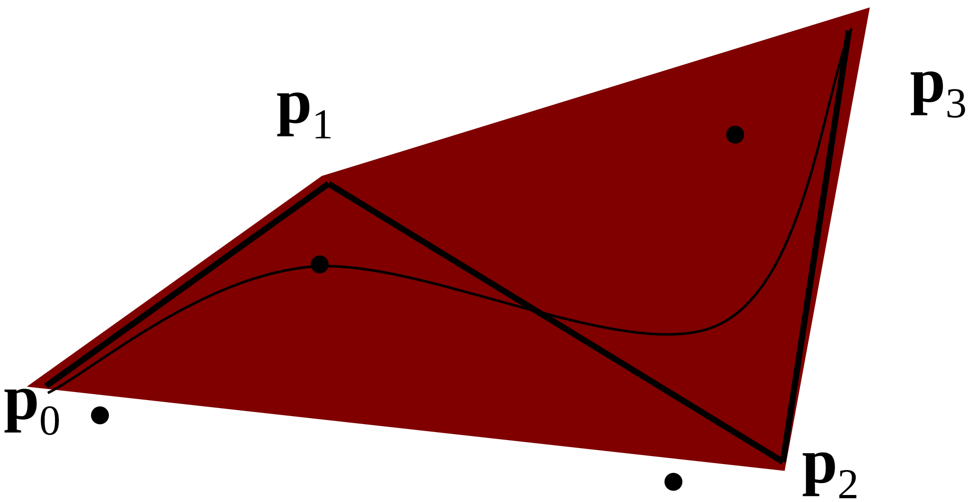


Useful Bézier Curve Properties

- Convex Hull property
- Affine Invariance

Convex Hull Property

- A Bézier curve is always inside the convex hull
  - Makes curve predictable
  - Allows culling, intersection testing, adaptive tessellation



Affine Invariance

Transforming Bézier curves

- Two ways to transform:
  - First transform control points, then compute spline points
  - First compute spline points, then transform them
- Results are identical
  - Invariant under affine transformations

Cubic Polynomial Form

Start with Bernstein form:

$$\mathbf{x}(t) = \left(-t^3 + 3t^2 - 3t + 1\right)\mathbf{p}_0 + \left(3t^3 - 6t^2 + 3t\right)\mathbf{p}_1 + \left(-3t^3 + 3t^2\right)\mathbf{p}_2 + \left(t^3\right)\mathbf{p}_3$$

Regroup into coefficients of t :

$$\mathbf{x}(t) = \left(-\mathbf{p}_0 + 3\mathbf{p}_1 - 3\mathbf{p}_2 + \mathbf{p}_3\right)t^3 + \left(3\mathbf{p}_0 - 6\mathbf{p}_1 + 3\mathbf{p}_2\right)t^2 + \left(-3\mathbf{p}_0 + 3\mathbf{p}_1\right)t + \left(\mathbf{p}_0\right)1$$

$$\boxed{\mathbf{x}(t) = \mathbf{a}t^3 + \mathbf{b}t^2 + \mathbf{c}t + \mathbf{d} \qquad \begin{aligned} \mathbf{a} &= \left(-\mathbf{p}_0 + 3\mathbf{p}_1 - 3\mathbf{p}_2 + \mathbf{p}_3\right) \\ \mathbf{b} &= \left(3\mathbf{p}_0 - 6\mathbf{p}_1 + 3\mathbf{p}_2\right) \\ \mathbf{c} &= \left(-3\mathbf{p}_0 + 3\mathbf{p}_1\right) \\ \mathbf{d} &= \left(\mathbf{p}_0\right) \end{aligned}}$$

- Good for fast evaluation
  - Precompute constant coefficients (**a**,**b**,**c**,**d**)
- Not much geometric intuition

Cubic Matrix Form

$$\mathbf{x}(t) = \begin{bmatrix} \vec{\mathbf{a}} & \vec{\mathbf{b}} & \vec{\mathbf{c}} & \mathbf{d} \end{bmatrix} \begin{bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}$$

$$\vec{\mathbf{a}} = \left(-\mathbf{p}_0 + 3\mathbf{p}_1 - 3\mathbf{p}_2 + \mathbf{p}_3\right)$$
$$\vec{\mathbf{b}} = \left(3\mathbf{p}_0 - 6\mathbf{p}_1 + 3\mathbf{p}_2\right)$$
$$\vec{\mathbf{c}} = \left(-3\mathbf{p}_0 + 3\mathbf{p}_1\right)$$
$$\mathbf{d} = \left(\mathbf{p}_0\right)$$

$$\mathbf{x}(t) = \underbrace{\begin{bmatrix} \mathbf{p}_0 & \mathbf{p}_1 & \mathbf{p}_2 & \mathbf{p}_3 \end{bmatrix}}_{\mathbf{G}_{Bez}} \underbrace{\begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}}_{\mathbf{B}_{Bez}} \underbrace{\begin{bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}}_{\mathbf{T}}$$

$$x(t) = G_{Bez}\, B_{Bez}\, T = C\, T$$

Matrix Form

- Other types of cubic splines use different basis matrices
- Efficient evaluation
  - Pre-compute **C**
  - Use existing 4x4 matrix hardware support

Lecture Overview

- Polynomial Curves
  - Introduction
  - Polynomial functions
- Bézier Curves
  - Introduction
  - Drawing Bézier curves
  - Piecewise Bézier curves

Drawing Bézier Curves

- Draw *line segments* or individual pixels
- Approximate the curve as a series of line segments (*tessellation*)
  - Uniform sampling
  - Adaptive sampling
  - Recursive subdivision

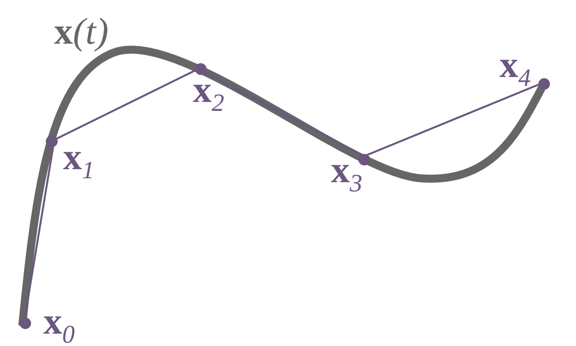
Uniform Sampling

- Approximate curve with N straight segments
  - N chosen in advance
  - Evaluate

$$\mathbf{x}_i = \mathbf{x}(t_i) \text{ where } t_i = \frac{i}{N} \text{ for } i = 0, 1, \ldots, N$$

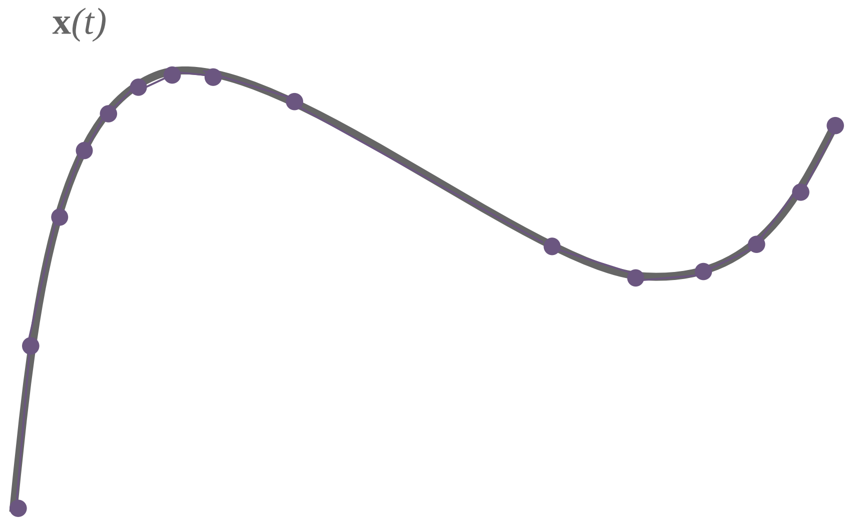
$$\mathbf{x}_i = \vec{\mathbf{a}}\frac{i^3}{N^3} + \vec{\mathbf{b}}\frac{i^2}{N^2} + \vec{\mathbf{c}}\frac{i}{N} + \mathbf{d}$$

  - Connect points with lines
- Too few points?
  - Poor approximation: “curve” is faceted
- Too many points?
  - Slow to draw too many line segments



Adaptive Sampling

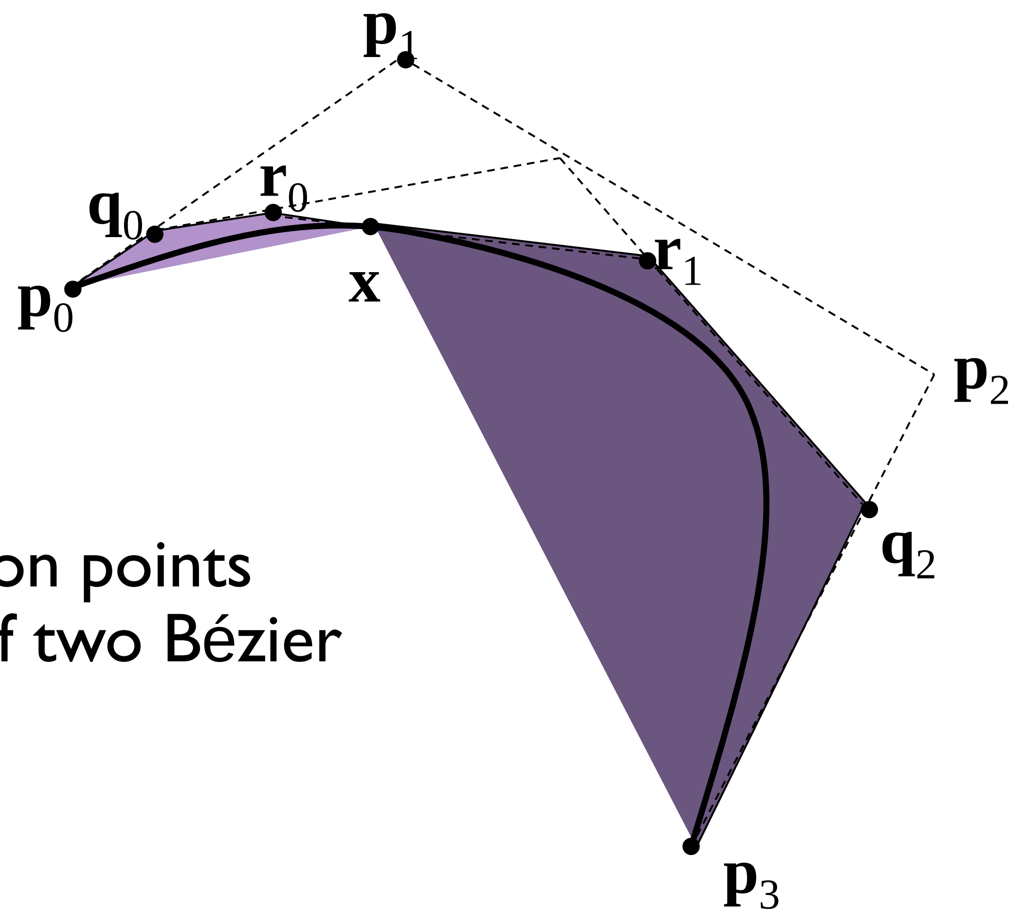
- Use only as many line segments as you need
  - Fewer segments where curve is mostly flat
  - More segments where curve bends
  - Segments never smaller than a pixel



Recursive Subdivision

- Any cubic curve segment can be expressed as a Bézier curve
- Any piece of a cubic curve is itself a cubic curve
- Therefore:
  - Any Bézier curve can be broken down into smaller Bézier curves

De Casteljau Subdivision



- De Casteljau construction points are the control points of two Bézier sub-segments

Adaptive Subdivision Algorithm

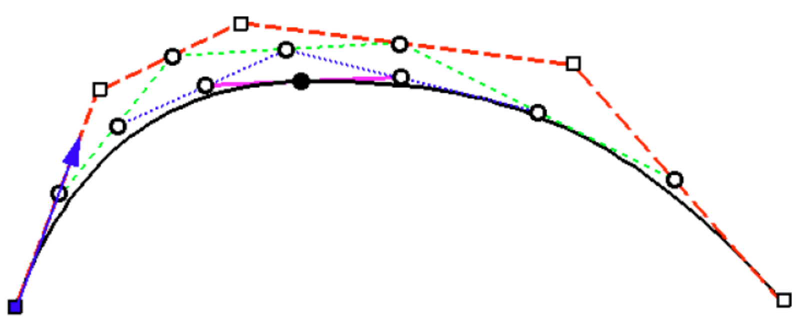
- Use De Casteljau construction to split Bézier segment in two
- For each part
  - If "flat enough": draw line segment
  - Else: continue recursion
- Curve is flat enough if hull is flat enough
  - Test how far the approximating control points are from a straight segment
    - If less than one pixel, the hull is flat enough

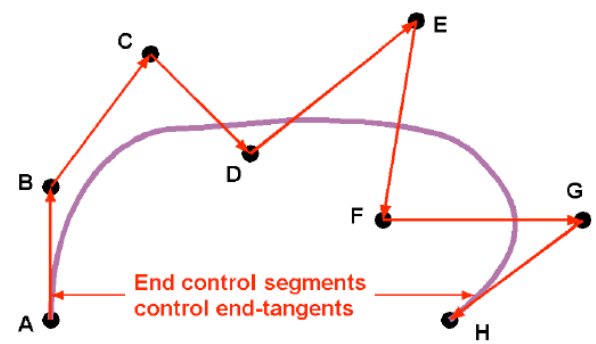
Lecture Overview

- Polynomial Curves
  - Introduction
  - Polynomial functions
- Bézier Curves
  - Introduction
  - Drawing Bézier curves
  - Longer curves

More Control Points

- Cubic Bézier curve limited to 4 control points
  - Cubic curve can only have one inflection (point where curve changes direction of bending)
  - Need more control points for more complex curves
- k-1 order Bézier curve with k control points

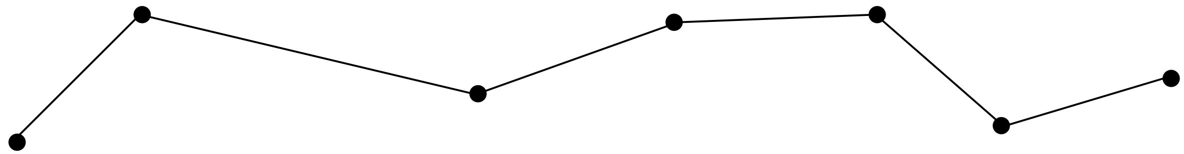




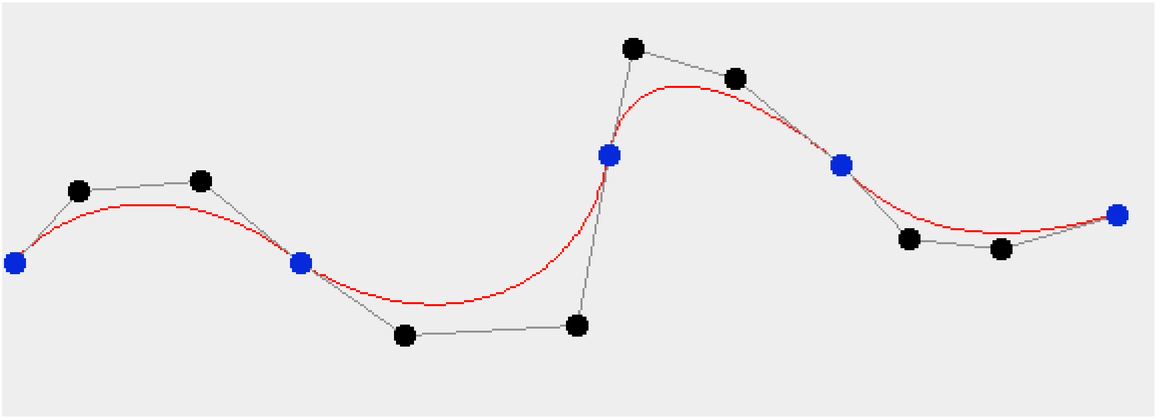
- Hard to control and hard to work with
  - Intermediate points don't have obvious effect on shape
  - Changing any control point changes the whole curve
  - Want *local support*: each control point only influences nearby portion of curve

Piecewise Curves

- Sequence of line segments
  - *Piecewise linear* curve



- Sequence of cubic curve segments
  - *Piecewise cubic* curve (here piecewise Bézier)



UCSD

Global Parameterization

- Given N curve segments $\mathbf{x}_0(t), \mathbf{x}_1(t), \ldots, \mathbf{x}_{N-1}(t)$
- Each is parameterized for t from 0 to 1
- Define a piecewise curve
  - Global parameter u from 0 to N

$$\mathbf{x}(u) = \begin{cases} \mathbf{x}_0(u), & 0 \le u \le 1 \\ \mathbf{x}_1(u-1), & 1 \le u \le 2 \\ \vdots & \vdots \\ \mathbf{x}_{N-1}(u-(N-1)), & N-1 \le u \le N \end{cases}$$

$$\mathbf{x}(u) = \mathbf{x}_i(u-i), \text{ where } i = \lfloor u \rfloor \quad (\text{and } \mathbf{x}(N) = \mathbf{x}_{N-1}(1))$$

- Alternate solution: u defined from 0 to 1

$$\mathbf{x}(u) = \mathbf{x}_i(Nu-i), \text{ where } i = \lfloor Nu \rfloor$$

UCSD

Piecewise Bézier curve

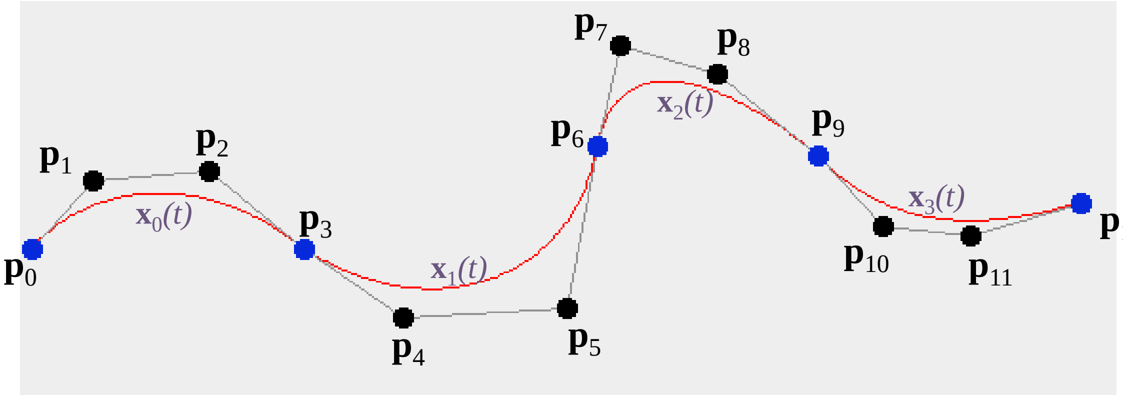
- Given $3N+1$ points $\mathbf{p}_0, \mathbf{p}_1, \ldots, \mathbf{p}_{3N}$
- Define N Bézier segments:

$$\mathbf{x}_0(t) = B_0(t)\mathbf{p}_0 + B_1(t)\mathbf{p}_1 + B_2(t)\mathbf{p}_2 + B_3(t)\mathbf{p}_3$$

$$\mathbf{x}_1(t) = B_0(t)\mathbf{p}_3 + B_1(t)\mathbf{p}_4 + B_2(t)\mathbf{p}_5 + B_3(t)\mathbf{p}_6$$

$$\vdots$$

$$\mathbf{x}_{N-1}(t) = B_0(t)\mathbf{p}_{3N-3} + B_1(t)\mathbf{p}_{3N-2} + B_2(t)\mathbf{p}_{3N-1} + B_3(t)\mathbf{p}_{3N}$$

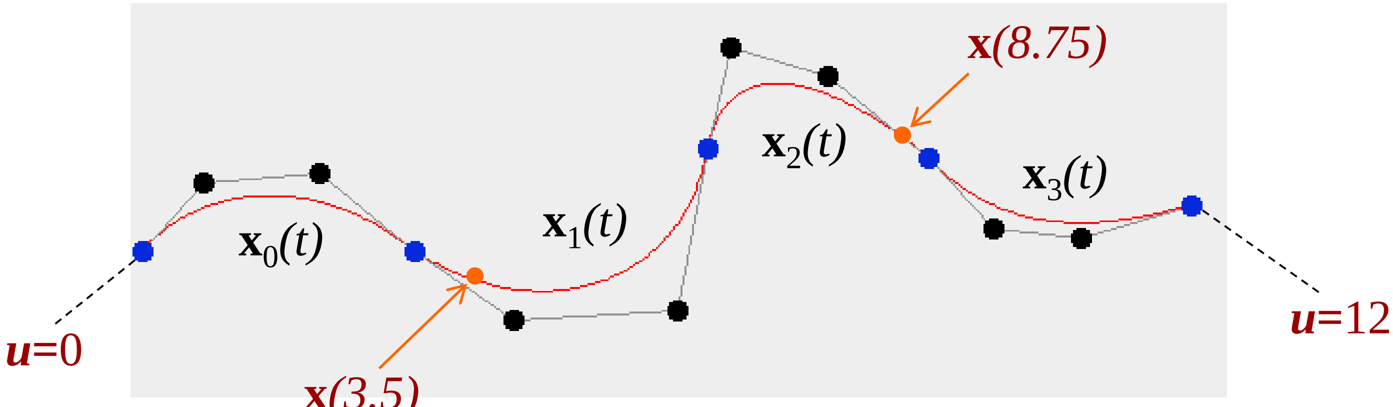


Piecewise Bézier Curve

- Parameter in $0 <= u <= 3N$

$$\mathbf{x}(u) = \begin{cases} \mathbf{x}_0(\frac{1}{3}u), & 0 \le u \le 3 \\ \mathbf{x}_1(\frac{1}{3}u - 1), & 3 \le u \le 6 \\ \vdots & \vdots \\ \mathbf{x}_{N-1}(\frac{1}{3}u - (N-1)), & 3N - 3 \le u \le 3N \end{cases}$$

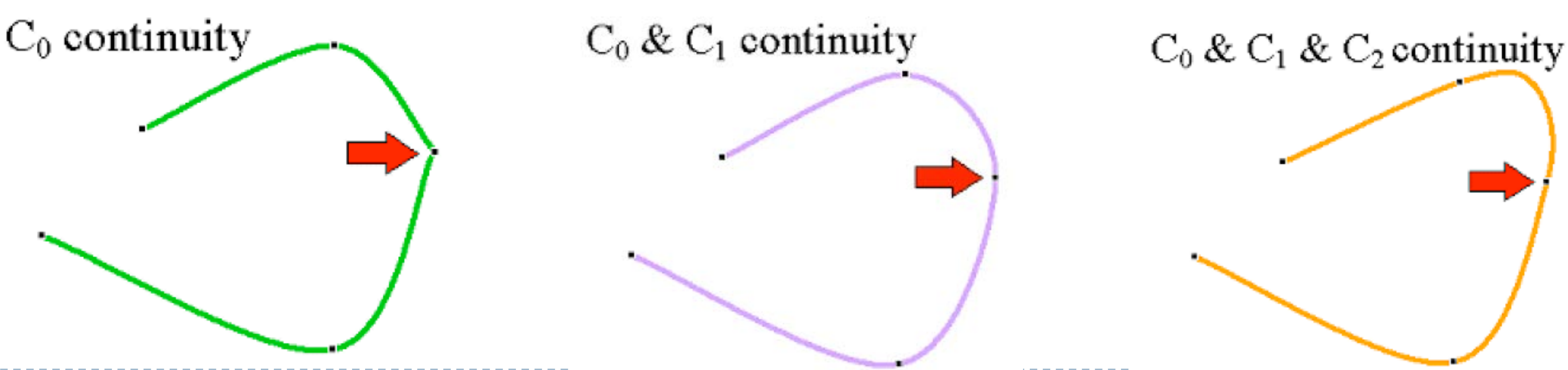
$\mathbf{x}(u) = \mathbf{x}_i\left(\frac{1}{3}u - i\right)$, where $i = \left\lfloor \frac{1}{3}u \right\rfloor$



UCSD

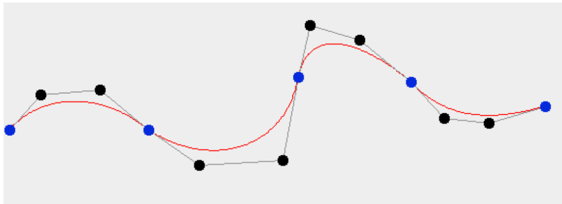
Parametric Continuity

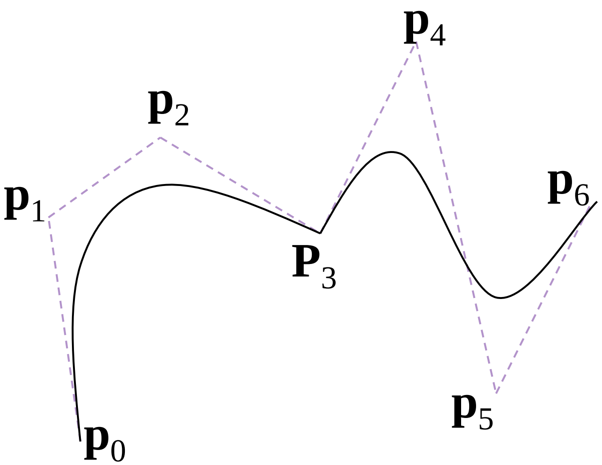
- C^0 continuity:
  - Curve segments are connected
- C^1 continuity:
  - C^0 & 1st-order derivatives agree
  - Curves have same tangents
  - Relevant for smooth shading
- C^2 continuity:
  - C^1 & 2nd-order derivatives agree
  - Curves have same tangents and curvature
  - Relevant for high quality reflections



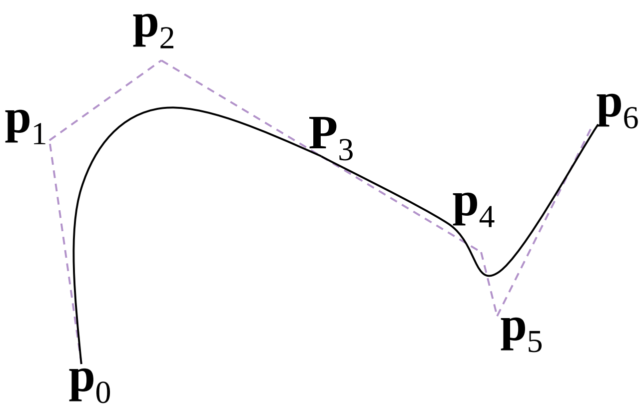
Piecewise Bézier Curve

- $3N+1$ points define N Bézier segments
- $\mathbf{x}(3i)=\mathbf{p}_{3i}$
- C_0 continuous by construction
- C_1 continuous at $\mathbf{p}_{3i}$ when $\mathbf{p}_{3i} - \mathbf{p}_{3i-1} = \mathbf{p}_{3i+1} - \mathbf{p}_{3i}$
- C_2 is harder to achieve and rarely necessary





C_1 discontinuous



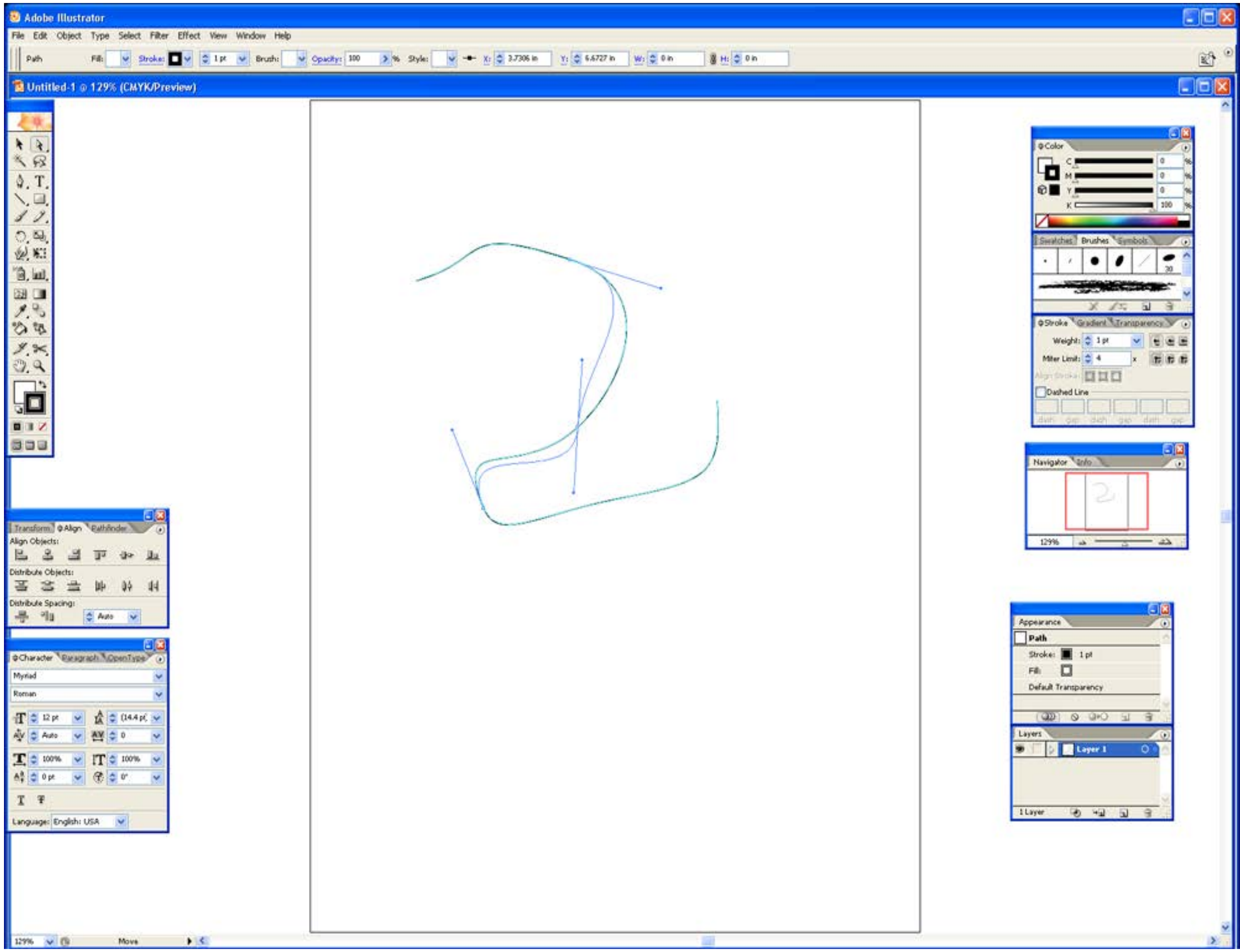
C_1 continuous

Piecewise Bézier Curves

- Used often in 2D drawing programs
- Inconveniences
  - Must have 4 or 7 or 10 or 13 or … (1 plus a multiple of 3) control points
  - Some points interpolate, others approximate
  - Need to impose constraints on control points to obtain C^1 continuity
- Solutions
  - User interface using "Bézier handles" to ascertain C^1 continuity
  - Generalization to B-splines or NURBS

Bézier Handles

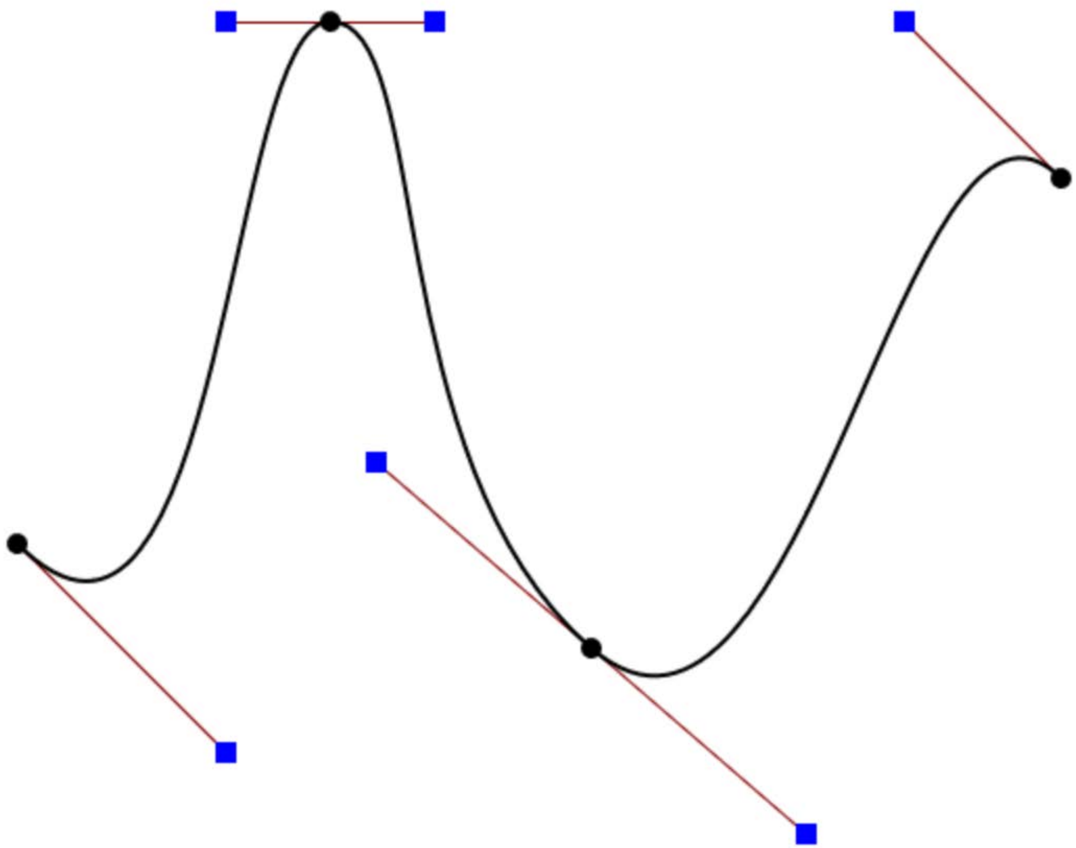
- Segment end points (interpolating) presented as curve control points
- Midpoints (approximating points) presented as "handles"
- Can have option to enforce C_1 continuity



Adobe Illustrator

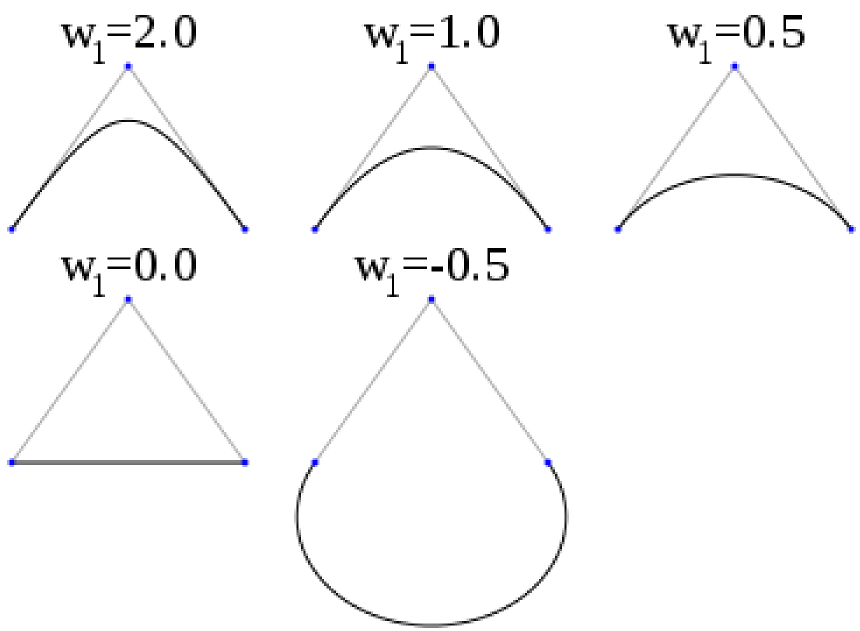
Demo: Bezier handles

- http://math.hws.edu/eck/cs424/notes2013/canvas/bezier.html



Rational Curves

- Weight causes point to "pull" more (or less)
- Can model circles with proper points and weights,
- Below: rational quadratic Bézier curve (three control points)



B-Splines

- B as in **B**asis-Splines
- Basis is blending function
- Difference to Bézier blending function:
  - B-spline blending function can be zero outside a particular range (limits scope over which a control point has influence)
- B-Spline is defined by control points and range in which each control point is active.

NURBS

- **N**on **U**niform **R**ational **B-S**plines
- Generalization of Bézier curves
- Non uniform:
- Combine B-Splines (limited scope of control points) and Rational Curves (weighted control points)
- Can exactly model conic sections (circles, ellipses)
- OpenGL support: see `gluNurbsCurve`
- Demos:
  - http://bentonian.com/teaching/AdvGraph0809/demos/Nurbs2d/index.html
  - http://geometrie.foretnik.net/files/NURBS-en.swf