

CSE 167:
Introduction to Computer Graphics
Lecture #2: Linear Algebra Primer

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Announcements

- ▶ Project I due next Friday at 2pm
 - ▶ Grading window is 2-3:30pm
 - ▶ Upload source code to TritonEd by 2pm
- ▶ 2nd discussion of project I Monday at 4pm

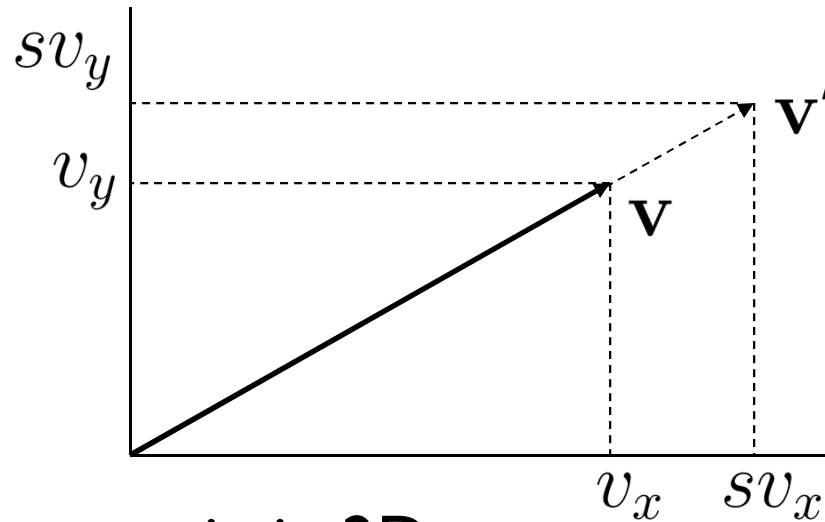
Lecture Overview

- ▶ **Affine Transformations**
- ▶ Homogeneous Coordinates

Affine Transformations

- ▶ Most important for graphics:
 - ▶ rotation, translation, scaling
- ▶ Wolfram MathWorld:
 - ▶ An **affine transformation** is any **transformation** that preserves collinearity (i.e., all points lying on a line initially still lie on a line after **transformation**) and ratios of distances (e.g., the midpoint of a line segment remains the midpoint after **transformation**).
- ▶ Implemented using matrix multiplications

Uniform Scale

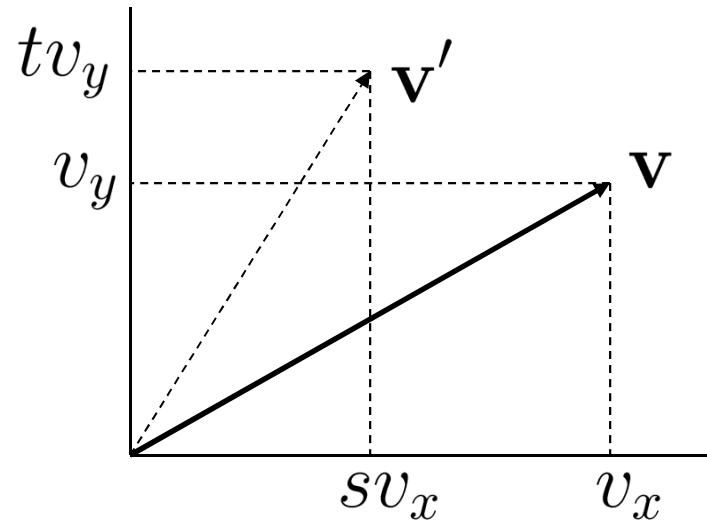


- ▶ Uniform scaling matrix in 2D

$$\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} \mathbf{v} = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} \begin{bmatrix} v_x \\ v_y \end{bmatrix} = \begin{bmatrix} v'_x \\ v'_y \end{bmatrix} = \mathbf{v}'$$

- ▶ Analogous in 3D

Non-Uniform Scale



- Nonuniform scaling matrix in 2D

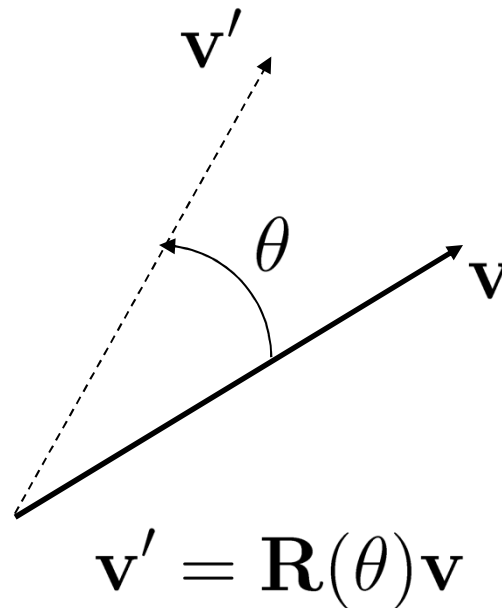
$$\begin{bmatrix} s & 0 \\ 0 & t \end{bmatrix} \mathbf{v} = \begin{bmatrix} s & 0 \\ 0 & t \end{bmatrix} \begin{bmatrix} v_x \\ v_y \end{bmatrix} = \begin{bmatrix} v'_x \\ v'_y \end{bmatrix} = \mathbf{v}'$$

- Analogous in 3D

Rotation in 2D

- ▶ Convention: positive angle rotates counterclockwise
- ▶ Rotation matrix

$$\mathbf{R}(\theta) = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$



Rotation in 3D

Rotation around coordinate axes

$$\mathbf{R}_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{bmatrix}$$

$$\mathbf{R}_y(\theta) = \begin{bmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{bmatrix}$$

$$\mathbf{R}_z(\theta) = \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Rotation in 3D

- ▶ Concatenation of rotations around x, y, z axes

$$\mathbf{R}_{x,y,z}(\theta_x, \theta_y, \theta_z) = \mathbf{R}_x(\theta_x)\mathbf{R}_y(\theta_y)\mathbf{R}_z(\theta_z)$$

- ▶ $\theta_x, \theta_y, \theta_z$ are called Euler angles
- ▶ Result depends on matrix order!

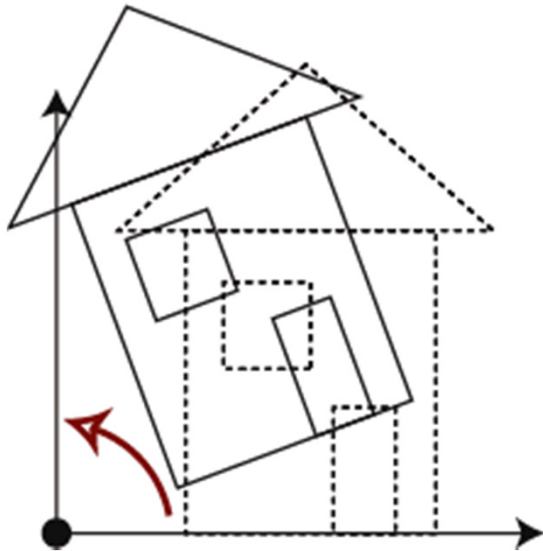
$$\mathbf{R}_x(\theta_x)\mathbf{R}_y(\theta_y)\mathbf{R}_z(\theta_z) \neq \mathbf{R}_z(\theta_z)\mathbf{R}_y(\theta_y)\mathbf{R}_x(\theta_x)$$

Rotation about an Arbitrary Axis

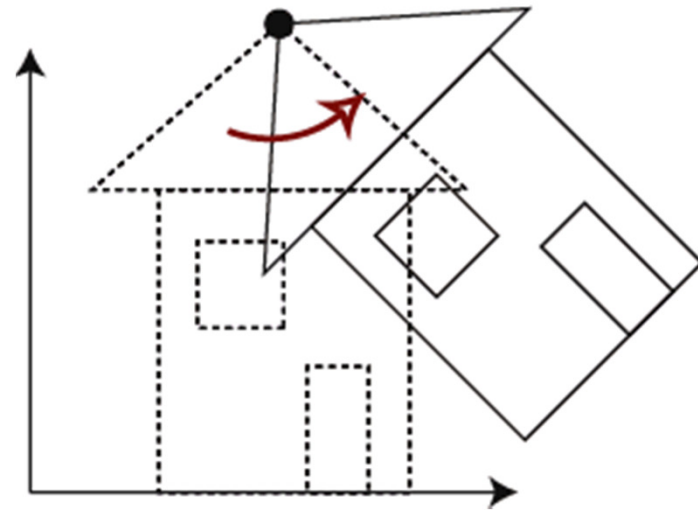
- ▶ Complicated!
- ▶ Rotate point $[x,y,z]$ about axis $[u,v,w]$ by angle θ :

$$\begin{bmatrix} \frac{u(ux+vy+wz)(1-\cos\theta) + (u^2+v^2+w^2)x\cos\theta + \sqrt{u^2+v^2+w^2}(-wy+ vz)\sin\theta}{u^2+v^2+w^2} \\ \frac{v(ux+vy+wz)(1-\cos\theta) + (u^2+v^2+w^2)y\cos\theta + \sqrt{u^2+v^2+w^2}(wx-uz)\sin\theta}{u^2+v^2+w^2} \\ \frac{w(ux+vy+wz)(1-\cos\theta) + (u^2+v^2+w^2)z\cos\theta + \sqrt{u^2+v^2+w^2}(-vx+uy)\sin\theta}{u^2+v^2+w^2} \end{bmatrix}$$

How to rotate around a Pivot Point?

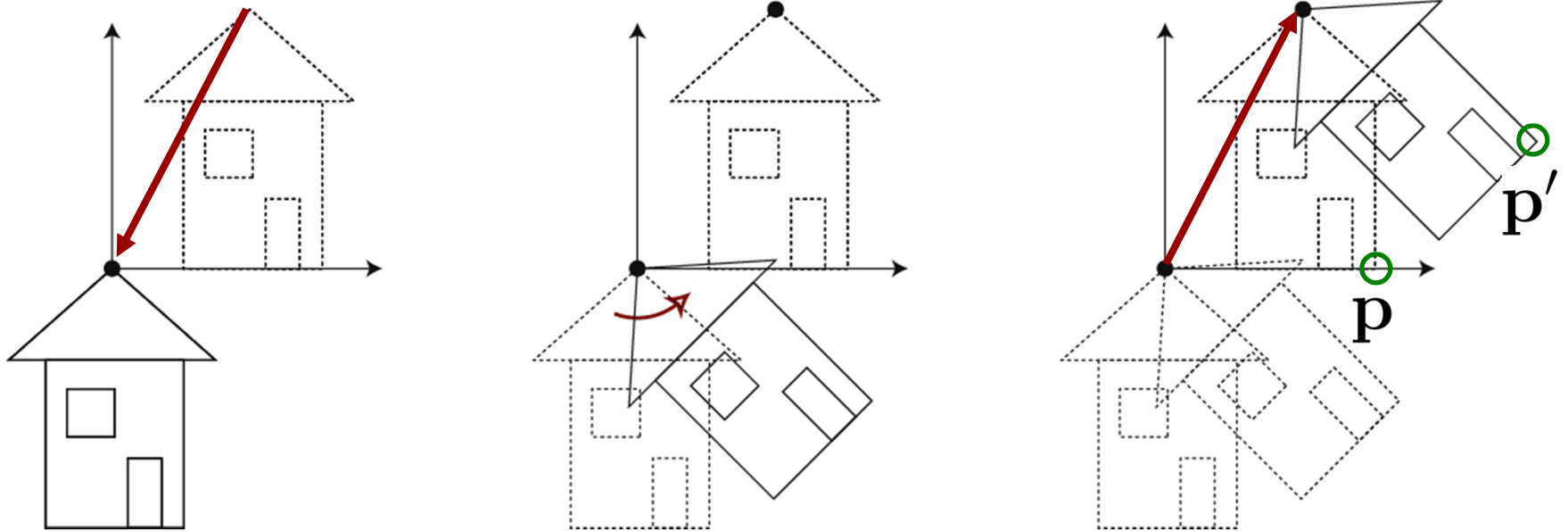


Rotation around
origin:
 $\mathbf{p}' = \mathbf{R} \mathbf{p}$



Rotation around
pivot point:
 $\mathbf{p}' = ?$

Rotating point p around a pivot point



1. Translation T 2. Rotation R 3. Translation T^{-1}

$$p' = T^{-1} R T p$$

Concatenating transformations

- ▶ Given a sequence of transformations $\mathbf{M}_3\mathbf{M}_2\mathbf{M}_1$

$$\mathbf{p}' = \mathbf{M}_3\mathbf{M}_2\mathbf{M}_1\mathbf{p}$$

$$\mathbf{M}_{total} = \mathbf{M}_3\mathbf{M}_2\mathbf{M}_1$$

$$\mathbf{p}' = \mathbf{M}_{total}\mathbf{p}$$

- ▶ Note: associativity applies:

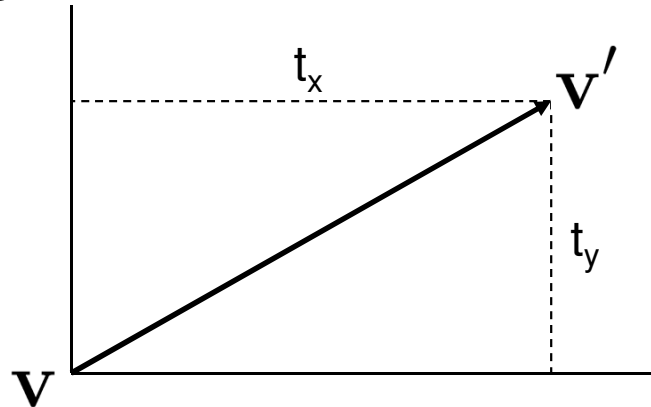
$$\mathbf{M}_{total} = (\mathbf{M}_3\mathbf{M}_2)\mathbf{M}_1 = \mathbf{M}_3(\mathbf{M}_2\mathbf{M}_1)$$

Lecture Overview

- ▶ Affine Transformations
- ▶ Homogeneous Coordinates

Translation

- ▶ Translation in 2D



- ▶ Translation matrix?

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

- ▶ Analogous in 3D: 4x4 matrix

Homogeneous Coordinates

- ▶ Basic: a trick to unify/simplify computations.
- ▶ Deeper: projective geometry
 - ▶ Interesting mathematical properties
 - ▶ Good to know, but less immediately practical
 - ▶ We will use some aspect of this when we do perspective projection

Homogeneous Coordinates

- ▶ Add an extra component. 1 for a point, 0 for a vector:

$$\mathbf{p} = \begin{bmatrix} p_x \\ p_y \\ p_z \\ 1 \end{bmatrix} \quad \vec{\mathbf{v}} = \begin{bmatrix} v_x \\ v_y \\ v_z \\ 0 \end{bmatrix}$$

- ▶ Combine **M** and **d** into single 4x4 matrix:

$$\begin{bmatrix} m_{xx} & m_{xy} & m_{xz} & d_x \\ m_{yx} & m_{yy} & m_{yz} & d_y \\ m_{zx} & m_{zy} & m_{zz} & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- ▶ And see what happens when we multiply...



Homogeneous Point Transform

- Transform a point:

$$\begin{bmatrix} p'_x \\ p'_y \\ p'_z \\ 1 \end{bmatrix} = \begin{bmatrix} m_{xx} & m_{xy} & m_{xz} & d_x \\ m_{yx} & m_{yy} & m_{yz} & d_y \\ m_{zx} & m_{zy} & m_{zz} & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ p_z \\ 1 \end{bmatrix} = \begin{bmatrix} m_{xx}p_x + m_{xy}p_y + m_{xz}p_z + d_x \\ m_{yx}p_x + m_{yy}p_y + m_{yz}p_z + d_y \\ m_{zx}p_x + m_{zy}p_y + m_{zz}p_z + d_z \\ 0 + 0 + 0 + 1 \end{bmatrix}$$
$$\mathbf{M} \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} + \vec{\mathbf{d}}$$

- Top three rows are the affine transform!
- Bottom row stays 1

Homogeneous Vector Transform

- Transform a vector:

$$\begin{bmatrix} v'_x \\ v'_y \\ v'_z \\ 0 \end{bmatrix} = \begin{bmatrix} m_{xx} & m_{xy} & m_{xz} & d_x \\ m_{yx} & m_{yy} & m_{yz} & d_y \\ m_{zx} & m_{zy} & m_{zz} & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_x \\ v_y \\ v_z \\ 0 \end{bmatrix} = \begin{bmatrix} m_{xx}v_x + m_{xy}v_y + m_{xz}v_z + 0 \\ m_{yx}v_x + m_{yy}v_y + m_{yz}v_z + 0 \\ m_{zx}v_x + m_{zy}v_y + m_{zz}v_z + 0 \\ 0 + 0 + 0 + 0 \end{bmatrix}$$



$$\mathbf{M} \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}$$

- Top three rows are the linear transform
 - Displacement **d** is properly ignored
- Bottom row stays 0

Homogeneous Arithmetic

- ▶ Legal operations always end in 0 or 1!

$$\text{vector+vector:} \quad \begin{bmatrix} \vdots \\ 0 \end{bmatrix} + \begin{bmatrix} \vdots \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} \vdots \\ 0 \end{bmatrix}$$

$$\text{vector-vector:} \quad \begin{bmatrix} \vdots \\ 0 \end{bmatrix} - \begin{bmatrix} \vdots \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} \vdots \\ 0 \end{bmatrix}$$

$$\text{scalar*vector:} \quad s \begin{bmatrix} \vdots \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} \vdots \\ 0 \end{bmatrix}$$

$$\text{point+vector:} \quad \begin{bmatrix} \vdots \\ 1 \end{bmatrix} + \begin{bmatrix} \vdots \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} \vdots \\ 1 \end{bmatrix}$$

$$\text{point-point:} \quad \begin{bmatrix} \vdots \\ 1 \end{bmatrix} - \begin{bmatrix} \vdots \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} \vdots \\ 0 \end{bmatrix}$$

$$\text{point+point:} \quad \begin{bmatrix} \vdots \\ 1 \end{bmatrix} + \begin{bmatrix} \vdots \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} \vdots \\ 2 \end{bmatrix}$$

$$\text{scalar*point:} \quad s \begin{bmatrix} \vdots \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} \vdots \\ s \end{bmatrix}$$

$$\left\{ \begin{array}{l} \text{weighted average} \\ \text{affine combination} \end{array} \right\} \text{ of points: } \frac{1}{3} \begin{bmatrix} \vdots \\ 1 \end{bmatrix} + \frac{2}{3} \begin{bmatrix} \vdots \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} \vdots \\ 1 \end{bmatrix}$$

Homogeneous Transforms

- ▶ Rotation, Scale, and Translation of points and vectors unified in a single matrix transformation:

$$\mathbf{p}' = \mathbf{M} \mathbf{p}$$

- ▶ Matrix has the form:
 - ▶ Last row always 0,0,0,1

$$\begin{bmatrix} m_{xx} & m_{xy} & m_{xz} & d_x \\ m_{yx} & m_{yy} & m_{yz} & d_y \\ m_{zx} & m_{zy} & m_{zz} & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- ▶ Transforms compose by matrix multiplication!
 - ▶ Same caveat: order of operations is important
 - ▶ Same note: Transforms operate right-to-left