CSE 167:
Introduction to Computer Graphics
Lecture #2: Linear Algebra Primer

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Announcements

- Project 1 due next Friday at 2pm
  - Grading window is 2-3:30pm
  - Upload source code to TritonEd by 2pm
- 2nd discussion of project 1 Monday at 4pm

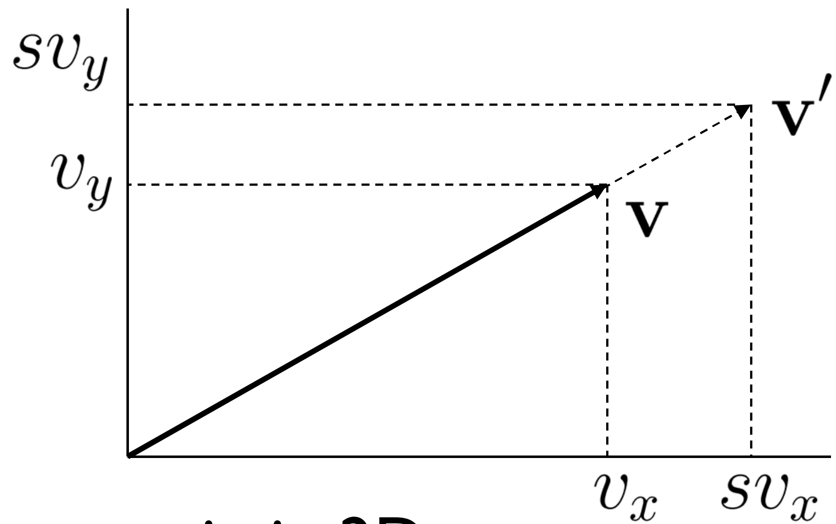
Lecture Overview

- Affine Transformations
- Homogeneous Coordinates

Affine Transformations

- Most important for graphics:
  - rotation, translation, scaling
- Wolfram MathWorld:
  - An **affine transformation** is any **transformation** that preserves collinearity (i.e., all points lying on a line initially still lie on a line after **transformation**) and ratios of distances (e.g., the midpoint of a line segment remains the midpoint after **transformation**).
- Implemented using matrix multiplications

Uniform Scale

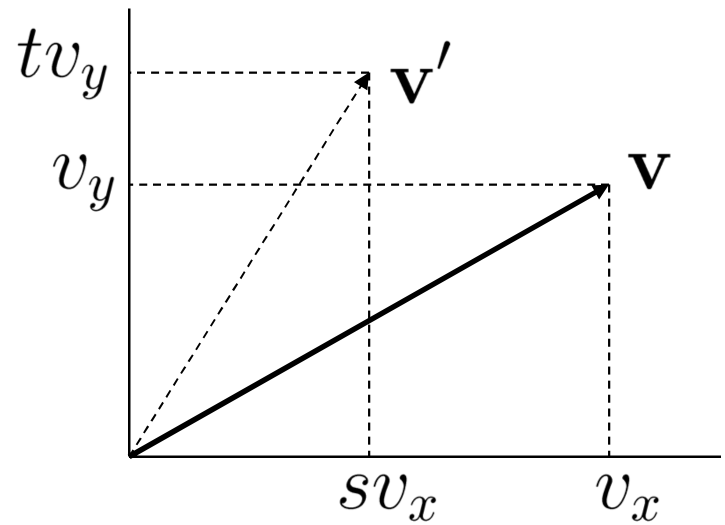


- Uniform scaling matrix in 2D

$$\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} \mathbf{v} = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} \begin{bmatrix} v_x \\ v_y \end{bmatrix} = \begin{bmatrix} v'_x \\ v'_y \end{bmatrix} = \mathbf{v}'$$

- Analogous in 3D

Non-Uniform Scale



- Nonuniform scaling matrix in 2D

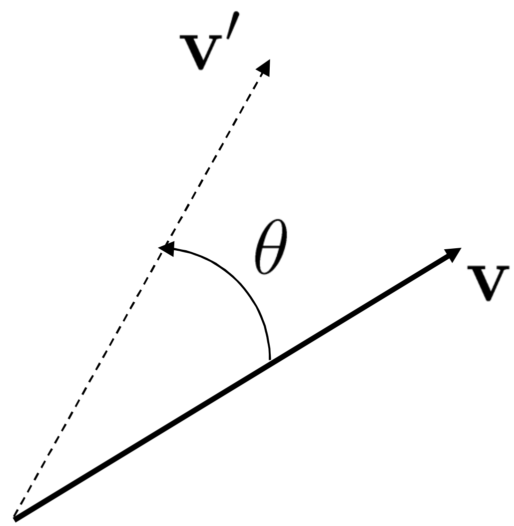
$$\begin{bmatrix} s & 0 \\ 0 & t \end{bmatrix} \mathbf{v} = \begin{bmatrix} s & 0 \\ 0 & t \end{bmatrix} \begin{bmatrix} v_x \\ v_y \end{bmatrix} = \begin{bmatrix} v'_x \\ v'_y \end{bmatrix} = \mathbf{v}'$$

- Analogous in 3D

Rotation in 2D

- Convention: positive angle rotates counterclockwise
- Rotation matrix

$$\mathbf{R}(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$



$$\mathbf{v}' = \mathbf{R}(\theta)\mathbf{v}$$

Rotation in 3D

Rotation around coordinate axes

$$\mathbf{R}_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

$$\mathbf{R}_y(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$\mathbf{R}_z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Rotation in 3D

- Concatenation of rotations around x, y, z axes

$$\mathbf{R}_{x,y,z}(\theta_x, \theta_y, \theta_z) = \mathbf{R}_x(\theta_x)\mathbf{R}_y(\theta_y)\mathbf{R}_z(\theta_z)$$

- $\theta_x, \theta_y, \theta_z$ are called Euler angles
- Result depends on matrix order!

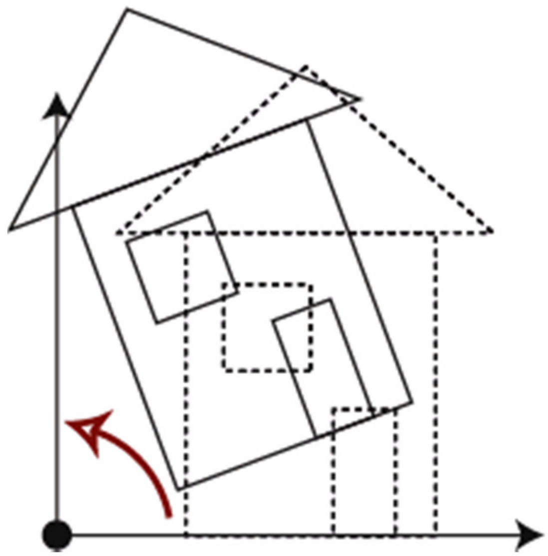
$$\mathbf{R}_x(\theta_x)\mathbf{R}_y(\theta_y)\mathbf{R}_z(\theta_z) \neq \mathbf{R}_z(\theta_z)\mathbf{R}_y(\theta_y)\mathbf{R}_x(\theta_x)$$

Rotation about an Arbitrary Axis

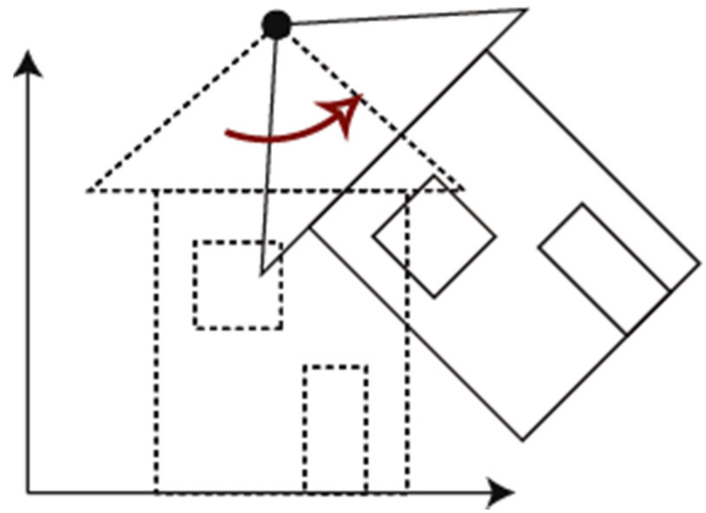
- Complicated!
- Rotate point [x,y,z] about axis [u,v,w] by angle θ:

$$\begin{bmatrix} \dfrac{u(ux+vy+wz)(1-\cos\theta)+(u^2+v^2+w^2)x\cos\theta+\sqrt{u^2+v^2+w^2}(-wy+vz)\sin\theta}{u^2+v^2+w^2} \\ \dfrac{v(ux+vy+wz)(1-\cos\theta)+(u^2+v^2+w^2)y\cos\theta+\sqrt{u^2+v^2+w^2}(wx-uz)\sin\theta}{u^2+v^2+w^2} \\ \dfrac{w(ux+vy+wz)(1-\cos\theta)+(u^2+v^2+w^2)z\cos\theta+\sqrt{u^2+v^2+w^2}(-vx+uy)\sin\theta}{u^2+v^2+w^2} \end{bmatrix}$$

How to rotate around a Pivot Point?

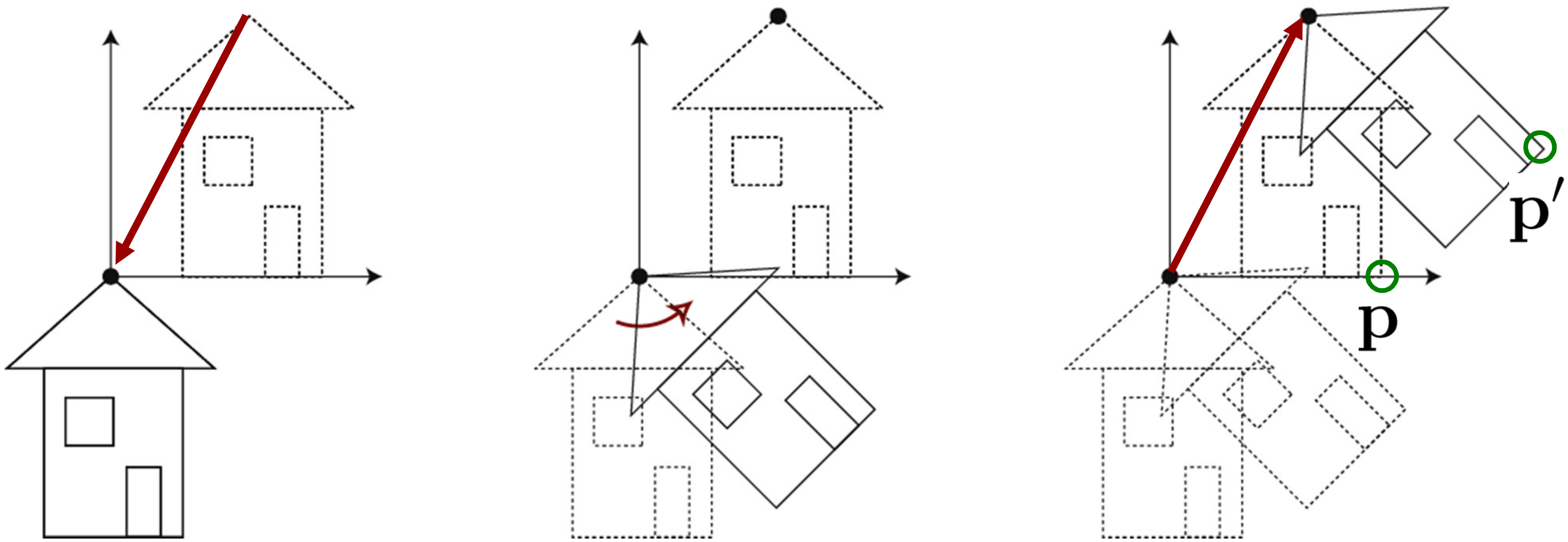


Rotation around origin:
p' = R p



Rotation around pivot point:
p' = ?

Rotating point p around a pivot point



1. Translation T 2. Rotation R 3. Translation T^{-1}

$$p' = T^{-1} R \, T \, p$$

Concatenating transformations

- Given a sequence of transformations $\mathbf{M}_3\mathbf{M}_2\mathbf{M}_1$

$$\mathbf{p}' = \mathbf{M}_3\mathbf{M}_2\mathbf{M}_1\mathbf{p}$$

$$\mathbf{M}_{total} = \mathbf{M}_3\mathbf{M}_2\mathbf{M}_1$$

$$\mathbf{p}' = \mathbf{M}_{total}\mathbf{p}$$

- Note: associativity applies:

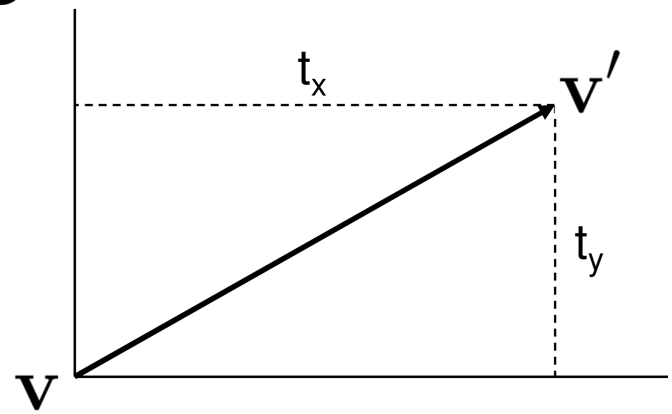
$$\mathbf{M}_{total} = (\mathbf{M}_3\mathbf{M}_2)\mathbf{M}_1 = \mathbf{M}_3(\mathbf{M}_2\mathbf{M}_1)$$

Lecture Overview

- Affine Transformations
- Homogeneous Coordinates

Translation

- Translation in 2D



- Translation matrix?

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

- Analogous in 3D: 4x4 matrix

Homogeneous Coordinates

- Basic: a trick to unify/simplify computations.
- Deeper: projective geometry
  - Interesting mathematical properties
  - Good to know, but less immediately practical
  - We will use some aspect of this when we do perspective projection

Homogeneous Coordinates

- Add an extra component. 1 for a point, 0 for a vector:

$$\mathbf{p} = \begin{bmatrix} p_x \\ p_y \\ p_z \\ 1 \end{bmatrix} \qquad \vec{\mathbf{v}} = \begin{bmatrix} v_x \\ v_y \\ v_z \\ 0 \end{bmatrix}$$

- Combine **M** and **d** into single 4x4 matrix:

$$\begin{bmatrix} m_{xx} & m_{xy} & m_{xz} & d_x \\ m_{yx} & m_{yy} & m_{yz} & d_y \\ m_{zx} & m_{zy} & m_{zz} & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- And see what happens when we multiply…

Homogeneous Point Transform

- Transform a point:

$$\begin{bmatrix} p'_x \\ p'_y \\ p'_z \\ 1 \end{bmatrix} = \begin{bmatrix} m_{xx} & m_{xy} & m_{xz} & d_x \\ m_{yx} & m_{yy} & m_{yz} & d_y \\ m_{zx} & m_{zy} & m_{zz} & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ p_z \\ 1 \end{bmatrix} = \begin{bmatrix} m_{xx}p_x + m_{xy}p_y + m_{xz}p_z + d_x \\ m_{yx}p_x + m_{yy}p_y + m_{yz}p_z + d_y \\ m_{zx}p_x + m_{zy}p_y + m_{zz}p_z + d_z \\ 0+0+0+1 \end{bmatrix}$$

$$\mathbf{M} \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} + \vec{\mathbf{d}}$$

- Top three rows are the affine transform!
- Bottom row stays 1

Homogeneous Vector Transform

- Transform a vector:

$$\begin{bmatrix} v'_x \\ v'_y \\ v'_z \\ 0 \end{bmatrix} = \begin{bmatrix} m_{xx} & m_{xy} & m_{xz} & d_x \\ m_{yx} & m_{yy} & m_{yz} & d_y \\ m_{zx} & m_{zy} & m_{zz} & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_x \\ v_y \\ v_z \\ 0 \end{bmatrix} = \begin{bmatrix} m_{xx}v_x + m_{xy}v_y + m_{xz}v_z + 0 \\ m_{yx}v_x + m_{yy}v_y + m_{yz}v_z + 0 \\ m_{zx}v_x + m_{zy}v_y + m_{zz}v_z + 0 \\ 0+0+0+0 \end{bmatrix}$$

$$\mathbf{M} \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}$$

- Top three rows are the linear transform
  - Displacement **d** is properly ignored
- Bottom row stays 0

Homogeneous Arithmetic

- Legal operations always end in 0 or 1!

$$\text{vector+vector:} \quad \begin{bmatrix} \vdots \\ 0 \end{bmatrix} + \begin{bmatrix} \vdots \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} \vdots \\ 0 \end{bmatrix}$$

$$\text{vector-vector:} \quad \begin{bmatrix} \vdots \\ 0 \end{bmatrix} - \begin{bmatrix} \vdots \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} \vdots \\ 0 \end{bmatrix}$$

$$\text{scalar*vector:} \quad s\begin{bmatrix} \vdots \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} \vdots \\ 0 \end{bmatrix}$$

$$\text{point+vector:} \quad \begin{bmatrix} \vdots \\ 1 \end{bmatrix} + \begin{bmatrix} \vdots \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} \vdots \\ 1 \end{bmatrix}$$

$$\text{point-point:} \quad \begin{bmatrix} \vdots \\ 1 \end{bmatrix} - \begin{bmatrix} \vdots \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} \vdots \\ 0 \end{bmatrix}$$

$$\text{point+point:} \quad \begin{bmatrix} \vdots \\ 1 \end{bmatrix} + \begin{bmatrix} \vdots \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} \vdots \\ 2 \end{bmatrix}$$

$$\text{scalar*point:} \quad s\begin{bmatrix} \vdots \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} \vdots \\ s \end{bmatrix}$$

$$\left\{ \begin{matrix} \text{weighted average} \\ \text{affine combination} \end{matrix} \right\} \text{of points:} \quad \frac{1}{3}\begin{bmatrix} \vdots \\ 1 \end{bmatrix} + \frac{2}{3}\begin{bmatrix} \vdots \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} \vdots \\ 1 \end{bmatrix}$$

Homogeneous Transforms

- Rotation, Scale, and Translation of points and vectors unified in a single matrix transformation:

$$\mathbf{p}' = \mathbf{M}\ \mathbf{p}$$

- Matrix has the form:
  - Last row always 0,0,0,1

$$\begin{bmatrix} m_{xx} & m_{xy} & m_{xz} & d_x \\ m_{yx} & m_{yy} & m_{yz} & d_y \\ m_{zx} & m_{zy} & m_{zz} & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

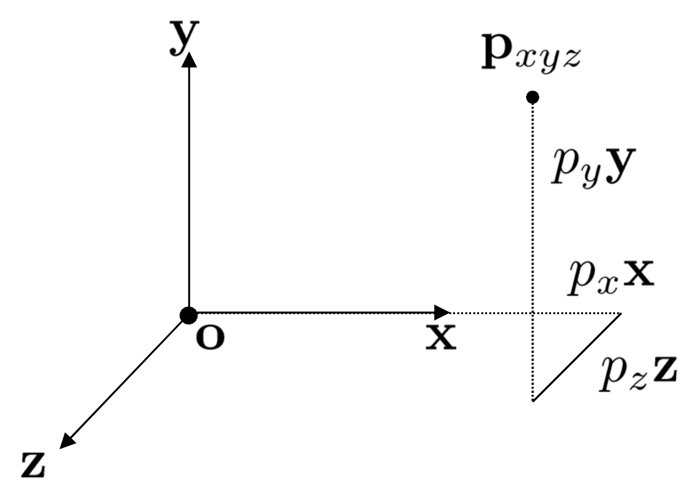
- Transforms compose by matrix multiplication!
  - Same caveat: order of operations is important
  - Same note: Transforms operate right-to-left

Lecture Overview

- Coordinate Transformation
- Typical Coordinate Systems
- Projection

Coordinate System

- Given point **p** in homogeneous coordinates: $\begin{bmatrix} p_x \\ p_y \\ p_z \\ 1 \end{bmatrix}$
- Coordinates describe the point's 3D position in a coordinate system with basis vectors **x**, **y**, **z** and origin **o**:

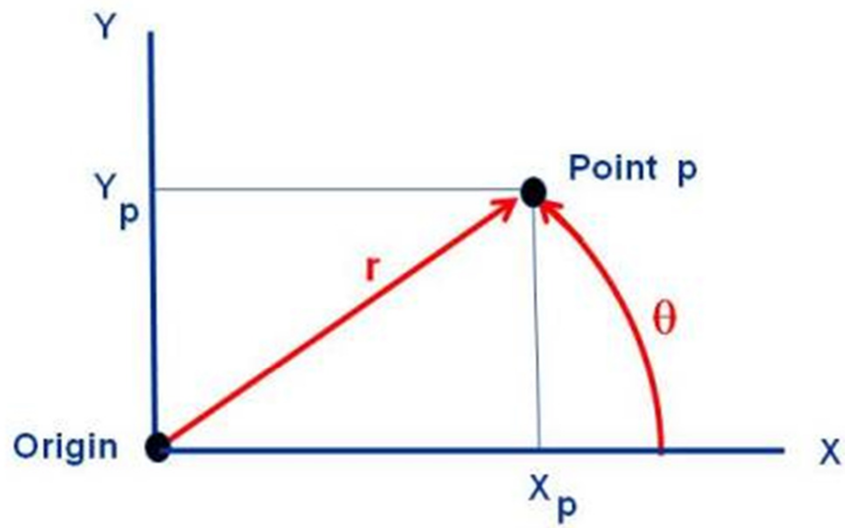


$$\mathbf{p}_{xyz} = p_x\mathbf{x} + p_y\mathbf{y} + p_z\mathbf{z} + \mathbf{o}$$

Rectangular and Polar Coordinates

National Aeronautics and Space Administration

Rectangular and Polar Coordinates



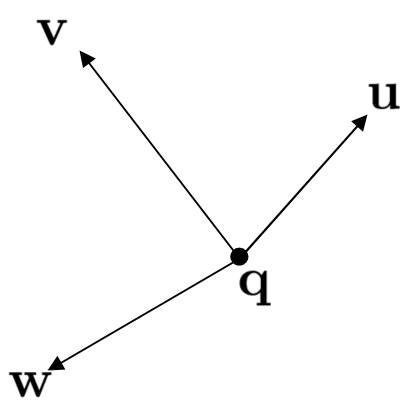
Point p can be located relative to the origin by Rectangular Coordinates (X_p, Y_p) or by Polar Coordinates (r, θ)

$$X_p = r \cos(\theta) \qquad r = \mathrm{sqrt}(X_p^2 + Y_p^2)$$

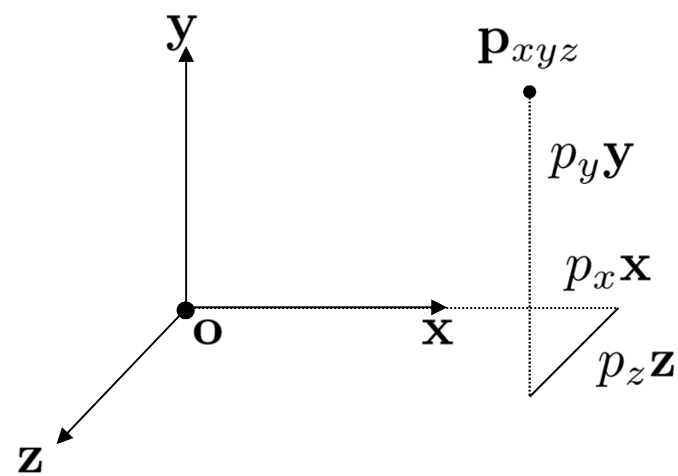
$$Y_p = r \sin(\theta) \qquad \theta = \tan^{-1}(Y_p / X_p)$$

www.nasa.gov

Coordinate Transformation



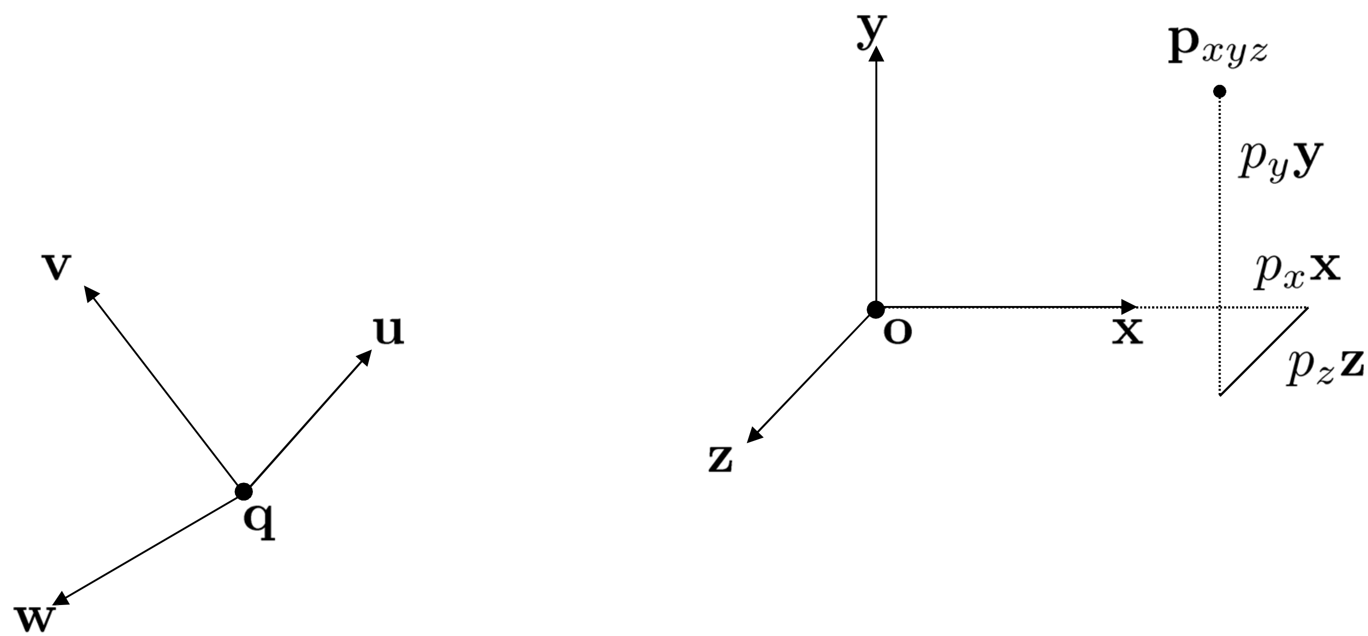
New **uvwq** coordinate system



Original **xyzo** coordinate system

Goal: Find coordinates of $\mathbf{p}_{xyz}$ in new **uvwq** coordinate system

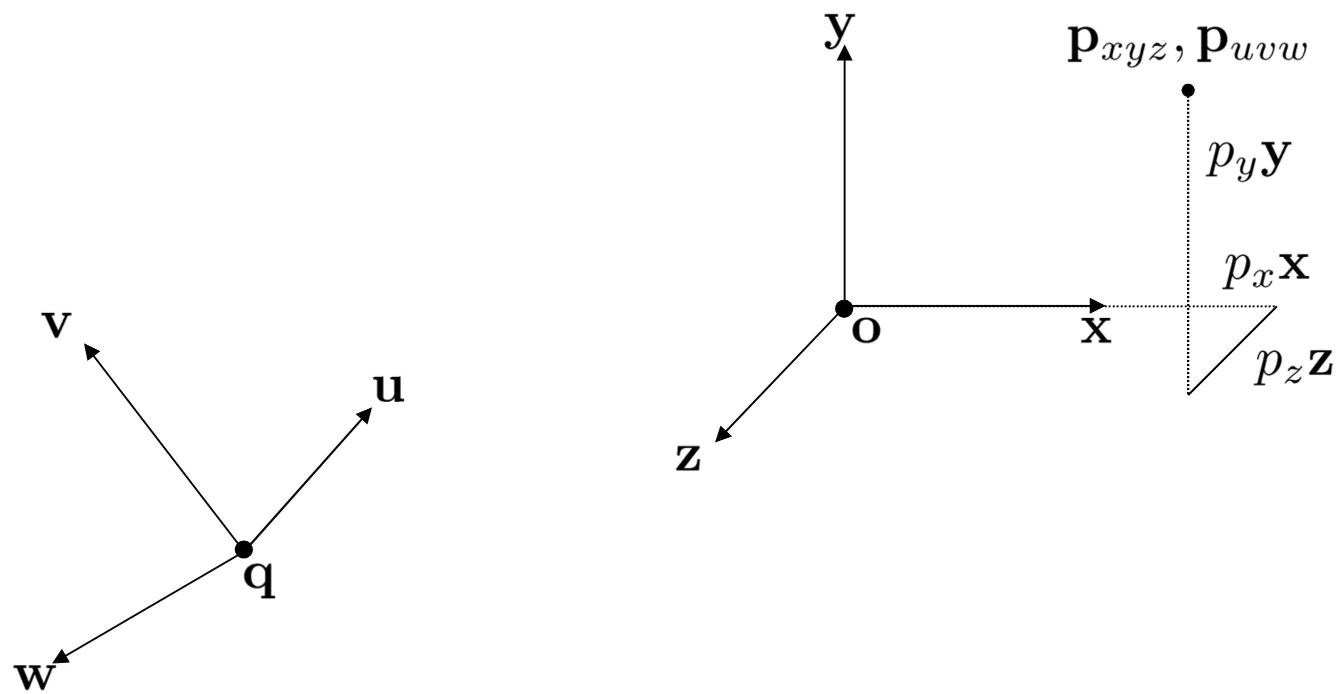
Coordinate Transformation



Express coordinates of **xyzo** reference frame with respect to **uvwq** reference frame:

$$\mathbf{x} = \begin{bmatrix} x_u \\ x_v \\ x_w \\ 0 \end{bmatrix} \qquad \mathbf{y} = \begin{bmatrix} y_u \\ y_v \\ y_w \\ 0 \end{bmatrix} \qquad \mathbf{z} = \begin{bmatrix} z_u \\ z_v \\ z_w \\ 0 \end{bmatrix} \qquad \mathbf{o} = \begin{bmatrix} o_u \\ o_v \\ o_w \\ 1 \end{bmatrix}$$

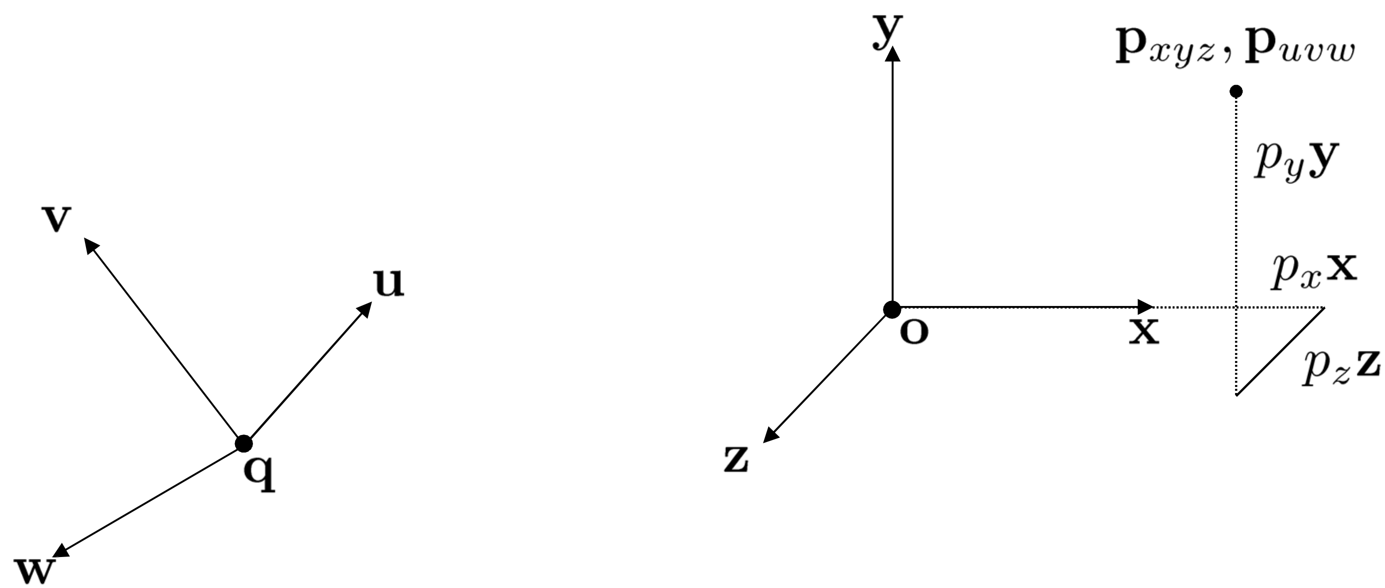
Coordinate Transformation



Point $\mathbf{p}$ expressed in new $\mathbf{uvwq}$ reference frame:

$$\mathbf{p}_{uvw} = p_x \begin{bmatrix} x_u \\ x_v \\ x_w \\ 0 \end{bmatrix} + p_y \begin{bmatrix} y_u \\ y_v \\ y_w \\ 0 \end{bmatrix} + p_z \begin{bmatrix} z_u \\ z_v \\ z_w \\ 0 \end{bmatrix} + \begin{bmatrix} o_u \\ o_v \\ o_w \\ 1 \end{bmatrix}$$

Coordinate Transformation



$$\mathbf{p}_{uvw} = \begin{bmatrix} x_u & y_u & z_u & o_u \\ x_v & y_v & z_v & o_v \\ x_w & y_w & z_w & o_w \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ p_z \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{x} & \mathbf{y} & \mathbf{z} & \mathbf{o} \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ p_z \\ 1 \end{bmatrix}$$

Coordinate Transformation

Inverse transformation

- Given point $\mathbf{P}_{uvw}$ w.r.t. reference frame **uvwq:**
  - Coordinates $\mathbf{P}_{xyz}$ w.r.t. reference frame **xyzo** are calculated as:

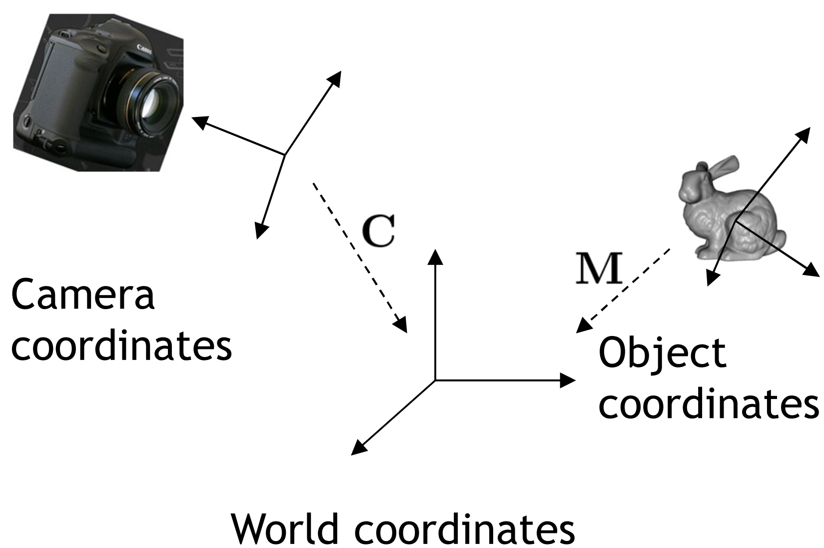
$$\mathbf{p}_{xyz} = \begin{bmatrix} x_u & y_u & z_u & o_u \\ x_v & y_v & z_v & o_v \\ x_w & y_w & z_w & o_w \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} p_u \\ p_v \\ p_w \\ 1 \end{bmatrix}$$

Lecture Overview

- Concatenating Transformations
- Coordinate Transformation
- Typical Coordinate Systems
- Projection

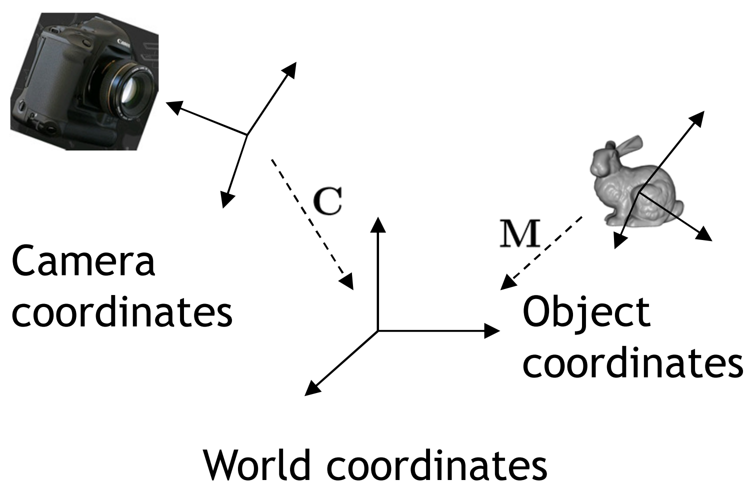
Typical Coordinate Systems

- In computer graphics, we typically use at least three coordinate systems:
  - World coordinate system
  - Camera coordinate system
  - Object coordinate system



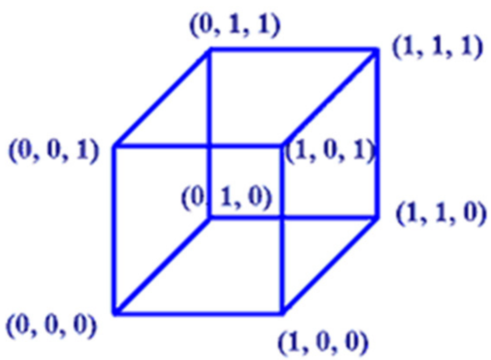
World Coordinates

- Common reference frame for all objects in the scene
- No standard for coordinate system orientation
  - If there is a ground plane, usually x/y is horizontal and z points up (height)
  - Otherwise, x/y is often screen plane, z points out of the screen

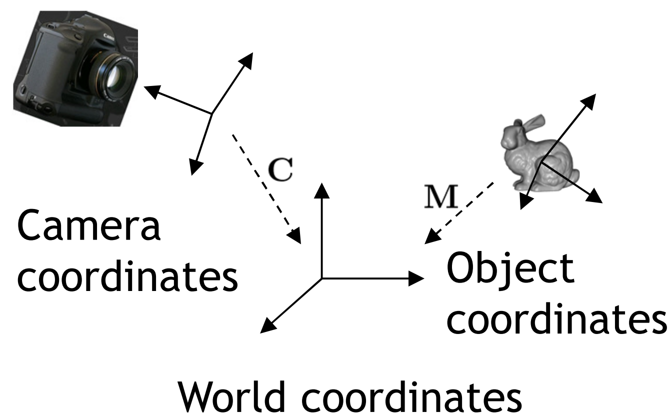


Object Coordinates

- Local coordinates in which points and other object geometry are given
- Often origin is in geometric center, on the base, or in a corner of the object
  - Depends on how object is generated or used.

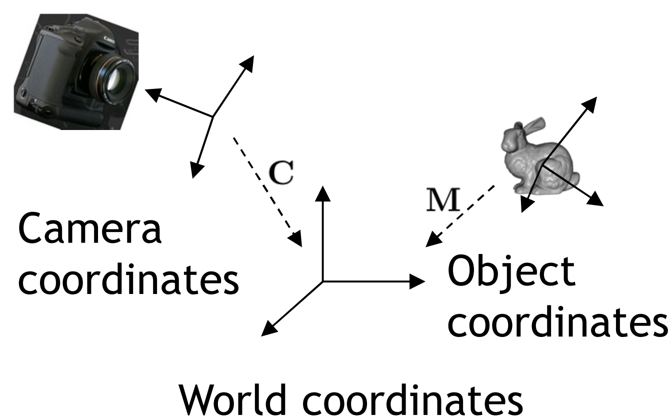


Source: http://motivate.maths.org



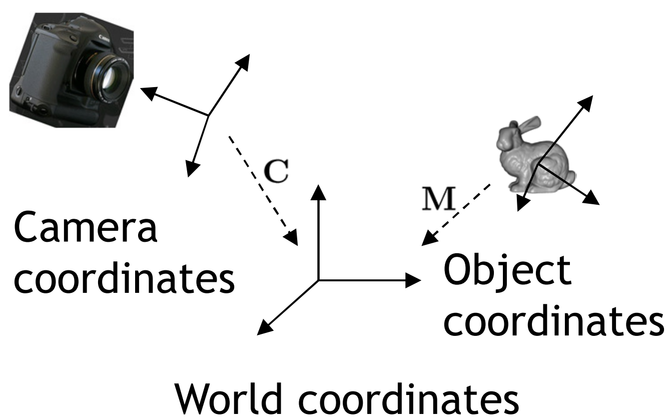
Object Transformation

- The transformation from object to world coordinates is different for each object.
- Defines placement of object in scene.
- Given by "model matrix" (model-to-world transformation) **M**.



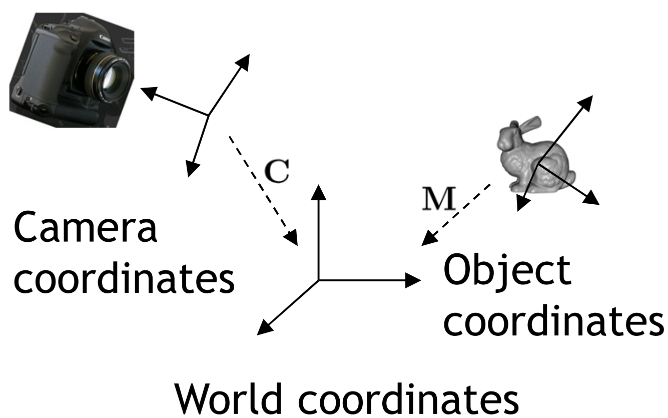
Camera Coordinate System

- Origin defines center of projection of camera
- x-y plane is parallel to image plane
- z-axis is perpendicular to image plane



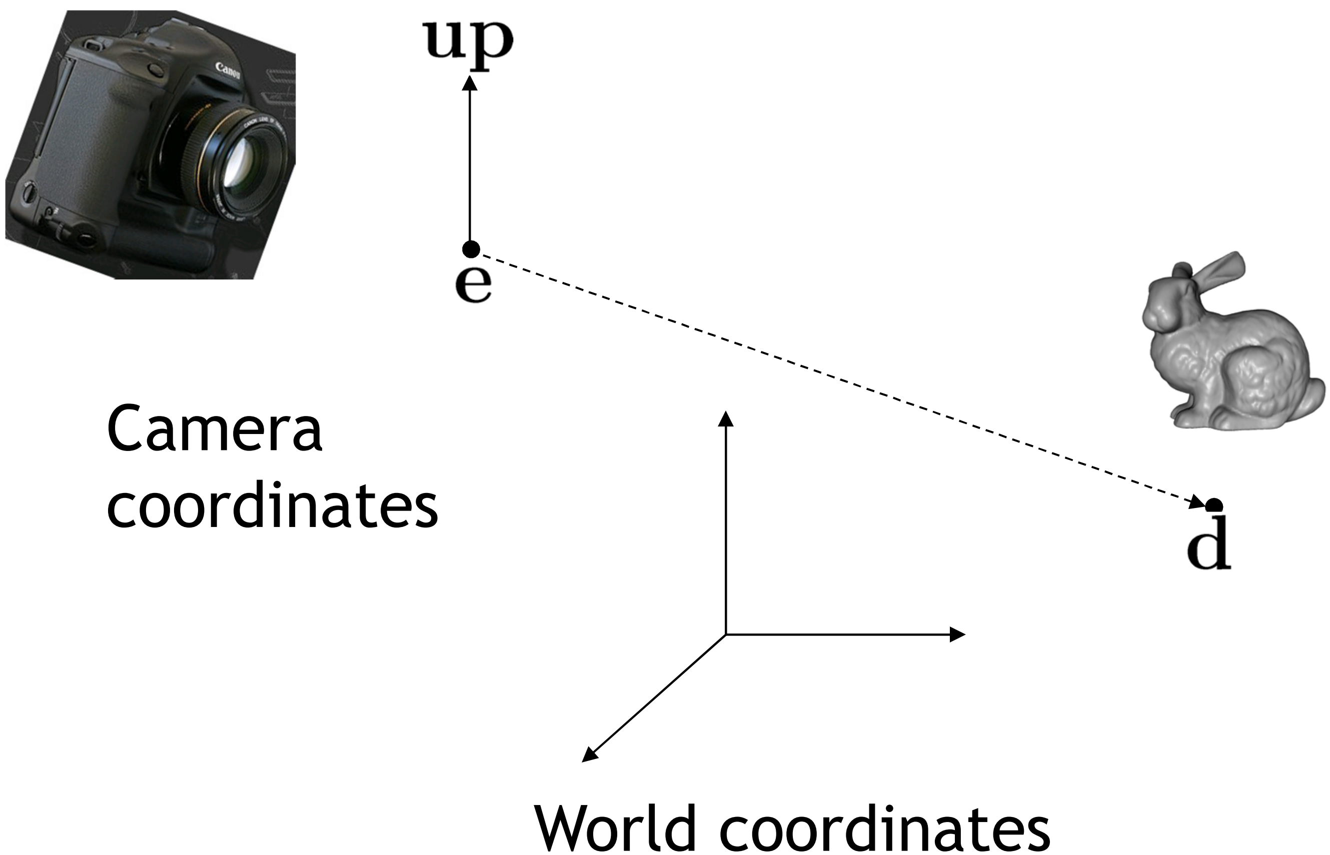
Camera Coordinate System

- The Camera Matrix defines the transformation from camera to world coordinates
  - Placement of camera in world



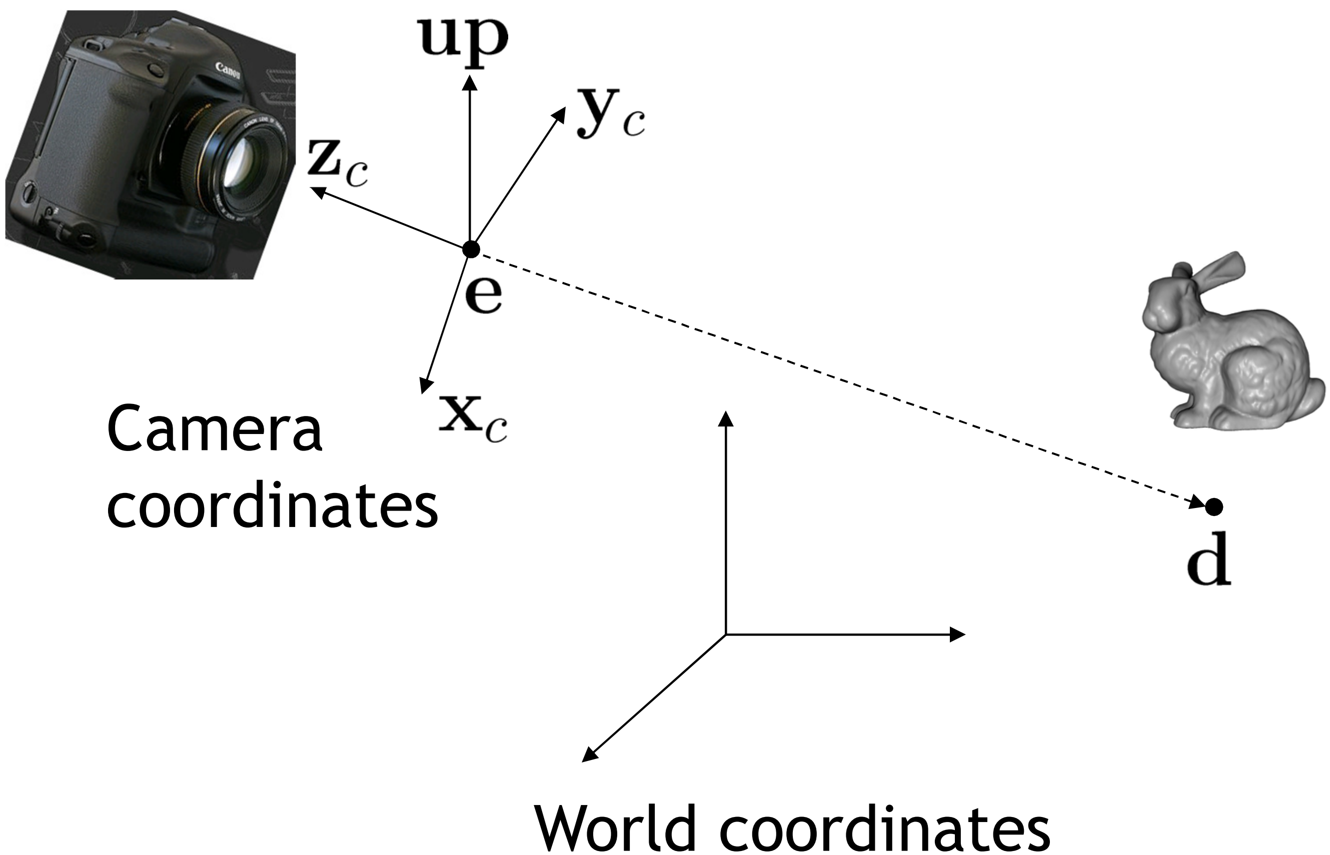
Camera Matrix

- Construct from center of projection **e**, look at **d**, up-vector **up:**



Camera Matrix

- Construct from center of projection **e**, look at **d**, up-vector **up** (up in camera coordinate system):



Camera Matrix

- z-axis

$$\boldsymbol{z}_C = \frac{\boldsymbol{e} - \boldsymbol{d}}{\|\boldsymbol{e} - \boldsymbol{d}\|}$$

- x-axis

$$\boldsymbol{x}_C = \frac{\boldsymbol{up} \times \boldsymbol{z}_C}{\|\boldsymbol{up} \times \boldsymbol{z}_C\|}$$

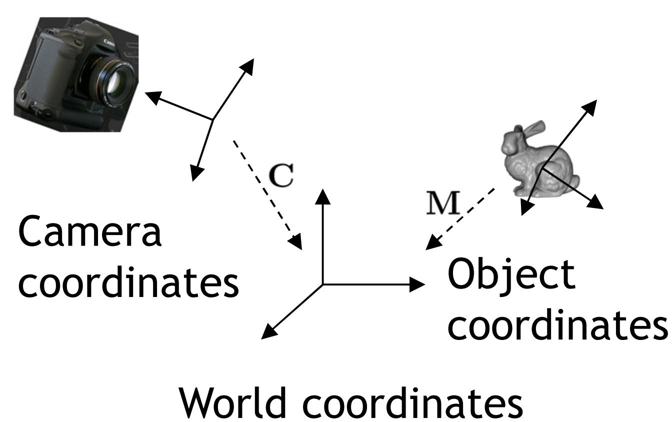
- y-axis

$$\boldsymbol{y}_C = \boldsymbol{z}_C \times \boldsymbol{x}_C = \frac{\boldsymbol{up}}{\|\boldsymbol{up}\|}$$

$$\boldsymbol{C} = \begin{bmatrix} \boldsymbol{x}_C & \boldsymbol{y}_C & \boldsymbol{z}_C & \boldsymbol{e} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Transforming Object to Camera Coordinates

- Object to world coordinates: **M**
- Camera to world coordinates: **C**
- Point to transform: **p**
- Resulting transformation equation: $\mathbf{p}' = \mathbf{C}^{-1}\,\mathbf{M}\,\mathbf{p}$



Tips for Notation

- Indicate coordinate systems with every point or matrix
  - Point: $\mathbf{p}_{object}$
  - Matrix: $\mathbf{M}_{object \rightarrow world}$
- Resulting transformation equation:

$$\mathbf{p}_{camera} = (\mathbf{C}_{camera \rightarrow world})^{-1}\ \mathbf{M}_{object \rightarrow world}\ \mathbf{p}_{object}$$

- Helpful hint: in source code use consistent names
  - Point: `p_object` or `p_obj` or `p_o`
  - Matrix: `object2world` or `obj2wld` or `o2w`
- Resulting transformation equation:

```
wld2cam = inverse(cam2wld);
p_cam = p_obj * obj2wld * wld2cam;
```

Inverse of Camera Matrix

- How to calculate the inverse of the camera matrix $\mathbf{C}^{-1}$?
- Generic matrix inversion is complex and compute-intensive
- Solution: affine transformation matrices can be inverted more easily
- Observation:
  - Camera matrix consists of translation and rotation: $\mathbf{T} \times \mathbf{R}$
- Inverse of rotation: $\mathbf{R}^{-1} = \mathbf{R}^{T}$
- Inverse of translation: $\mathbf{T}(t)^{-1} = \mathbf{T}(-t)$
- Inverse of camera matrix: $\mathbf{C}^{-1} = \mathbf{R}^{-1} \times \mathbf{T}^{-1}$

Objects in Camera Coordinates

- We have things lined up the way we like them on screen
  - **x** to the right
  - **y** up
  - **-z** into the screen
  - Objects to look at are in front of us, i.e. have negative z values
- But objects are still in 3D
- Next step: project scene to 2D plane

Lecture Overview

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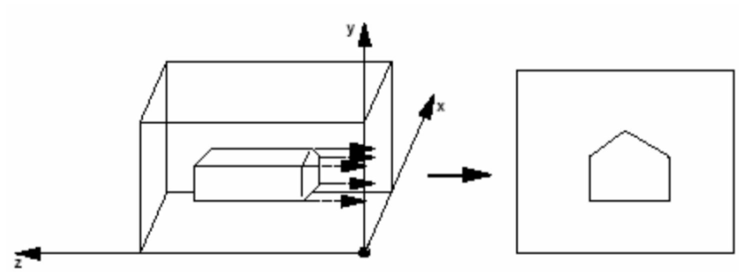
Projection

- Goal:
  Given 3D points (vertices) in camera coordinates, determine corresponding image coordinates
- Transforming 3D points into 2D is called Projection
- OpenGL supports two types of projection:
  - Orthographic Projection (=Parallel Projection)
  - Perspective Projection

Orthographic Projection

- Can be done by ignoring **z**-coordinate
  - Use camera space **xy** coordinates as image coordinates
- Project points to **x-y** plane along parallel lines



- Often used in graphical illustrations, architecture, 3D modeling

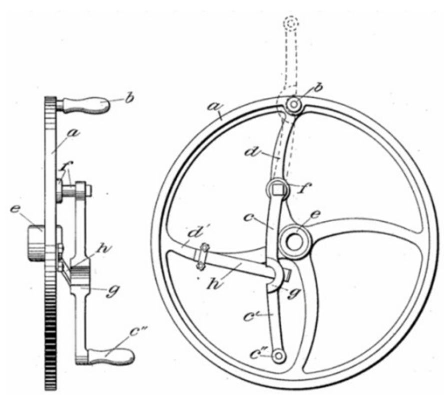
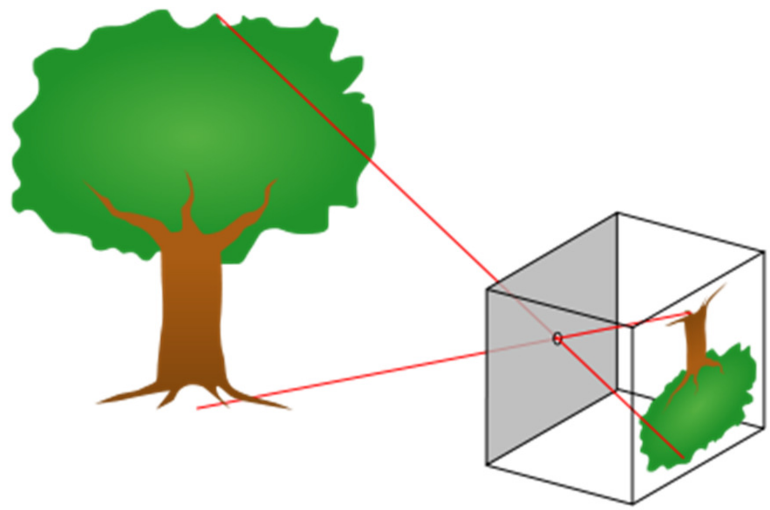


Fig. 2. Fig. 1.

Perspective Projection

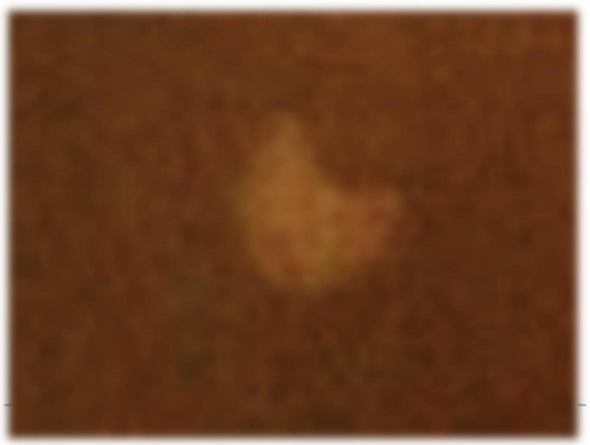
- Most common for computer graphics
- Simplified model of human eye, or camera lens (*pinhole camera*)



- Things farther away appear to be smaller
- Discovery attributed to Filippo Brunelleschi (Italian architect) in the early 1400's

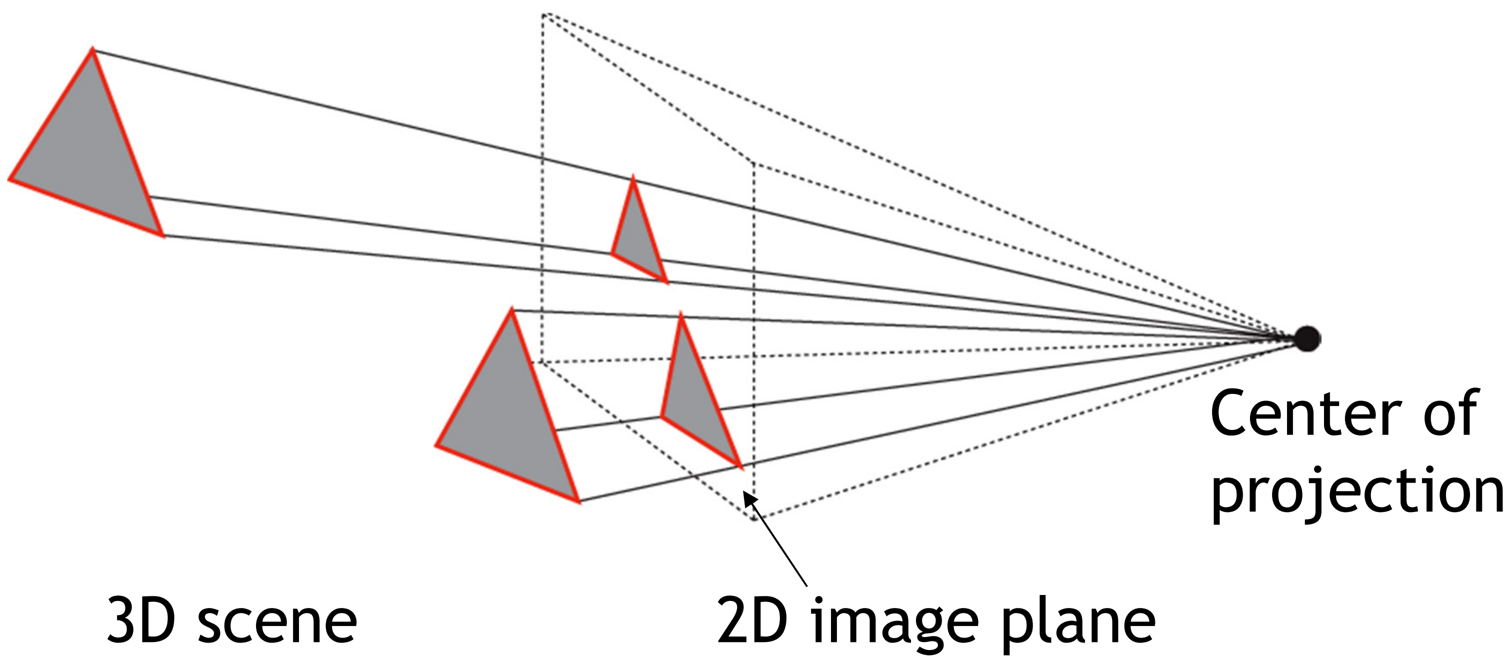
Pinhole Camera

- San Diego, May 20^{th}, 2012



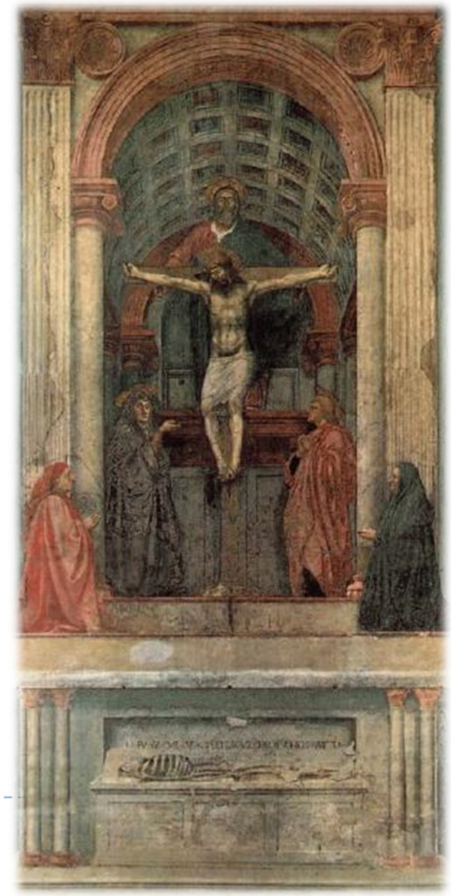
Perspective Projection

- Project along rays that converge in center of projection



Perspective Projection

Parallel lines are no longer parallel, converge in one point



Earliest example:
La Trinitá (1427) by Masaccio

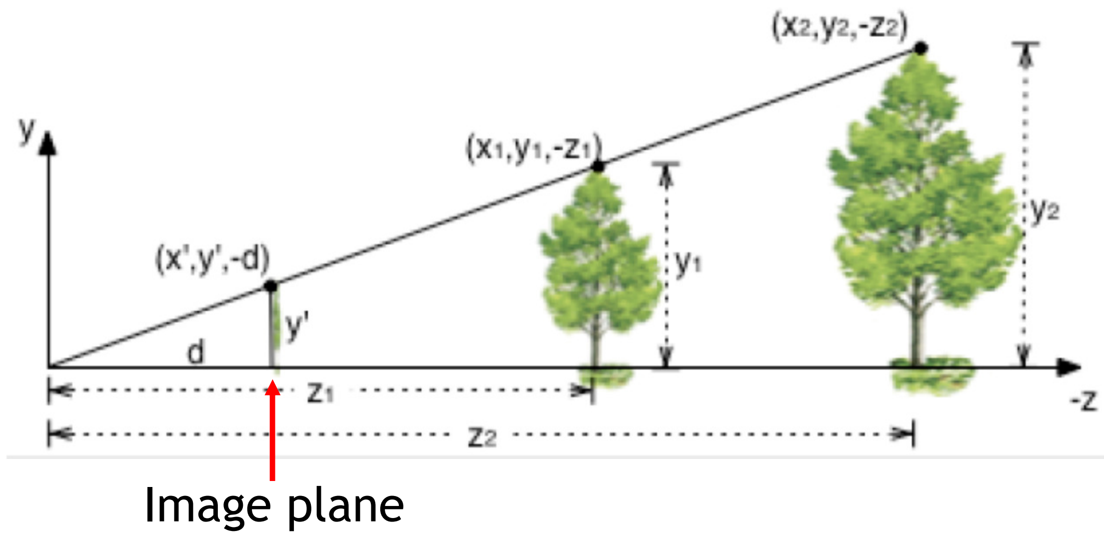
Perspective Projection

From law of ratios in similar triangles follows:

$$\frac{y'}{d} = \frac{y_1}{z_1} \rightarrow y' = \frac{y_1 d}{z_1}$$

Similarly: $$x' = \frac{x_1 d}{z_1}$$

By definition: $$z' = d$$



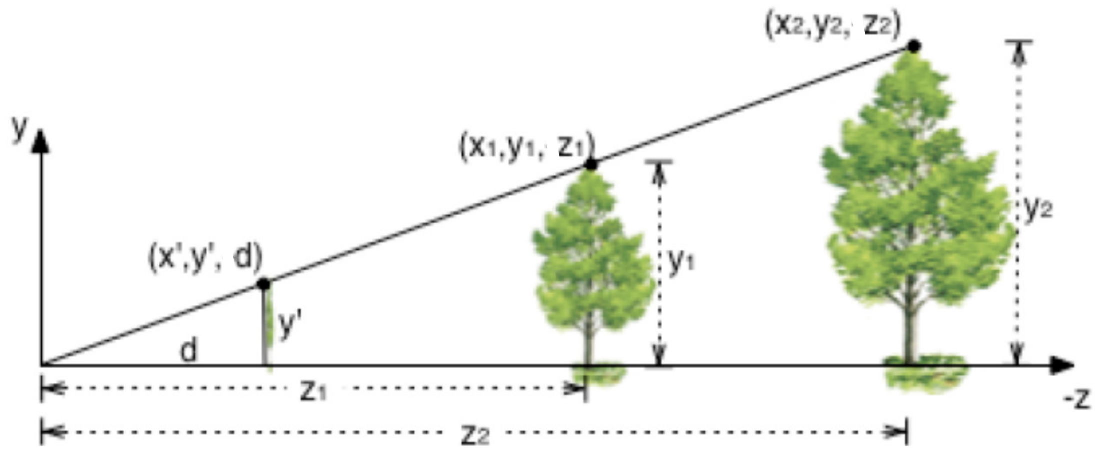
- We can express this using homogeneous coordinates and 4x4 matrices as follows

Perspective Projection

$$x' = \frac{x_1 d}{z_1}$$

$$y' = \frac{y_1 d}{z_1}$$

$$z' = d$$



$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1/d & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \\ z/d \end{bmatrix} \Rightarrow \begin{bmatrix} xd/z \\ yd/z \\ d \\ 1 \end{bmatrix}$$

Projection matrix Homogeneous division

Perspective Projection

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1/d & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \\ z/d \end{bmatrix} = \begin{bmatrix} xd/z \\ yd/z \\ d \\ 1 \end{bmatrix}$$

Projection matrix P

- Using projection matrix, homogeneous division seems more complicated than just multiplying all coordinates by d/z, so why do it?
- It will allow us to:
  - Handle different types of projections in a unified way
  - Define arbitrary view volumes

Lecture Overview

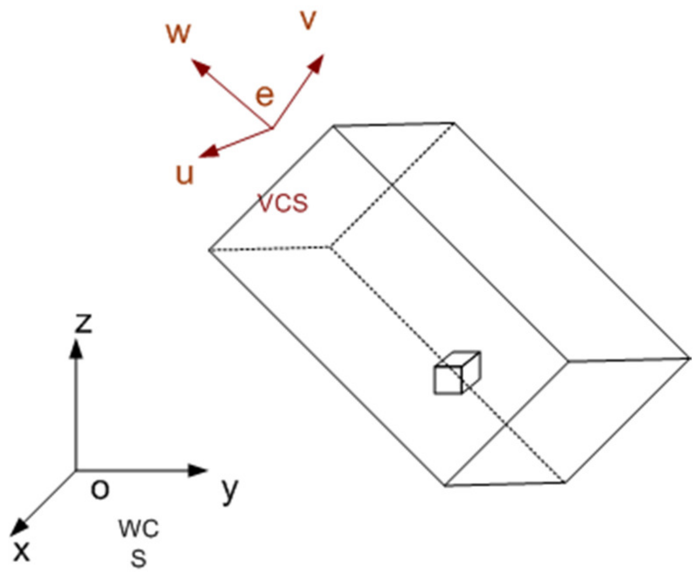
- View Volumes
- Vertex Transformation
- Rendering Pipeline
- Culling

View Volumes

- View volume = 3D volume seen by camera

Orthographic view volume

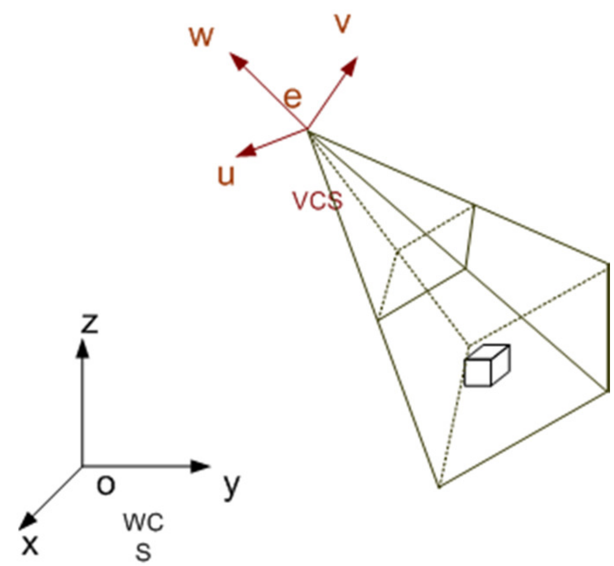
Camera coordinates



World coordinates

Perspective view volume

Camera coordinates



World coordinates

Projection Matrix

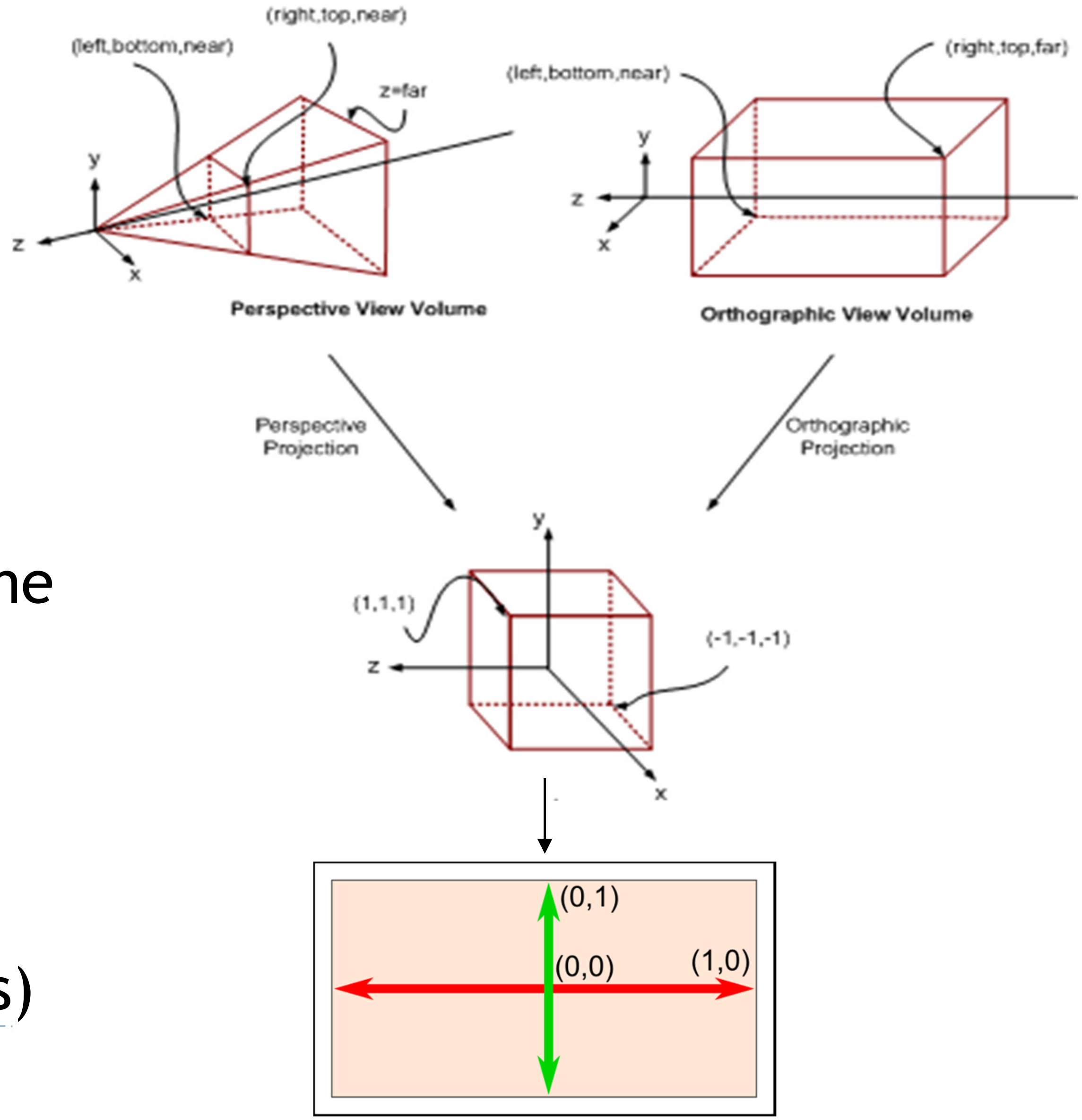
Camera coordinates

Projection matrix

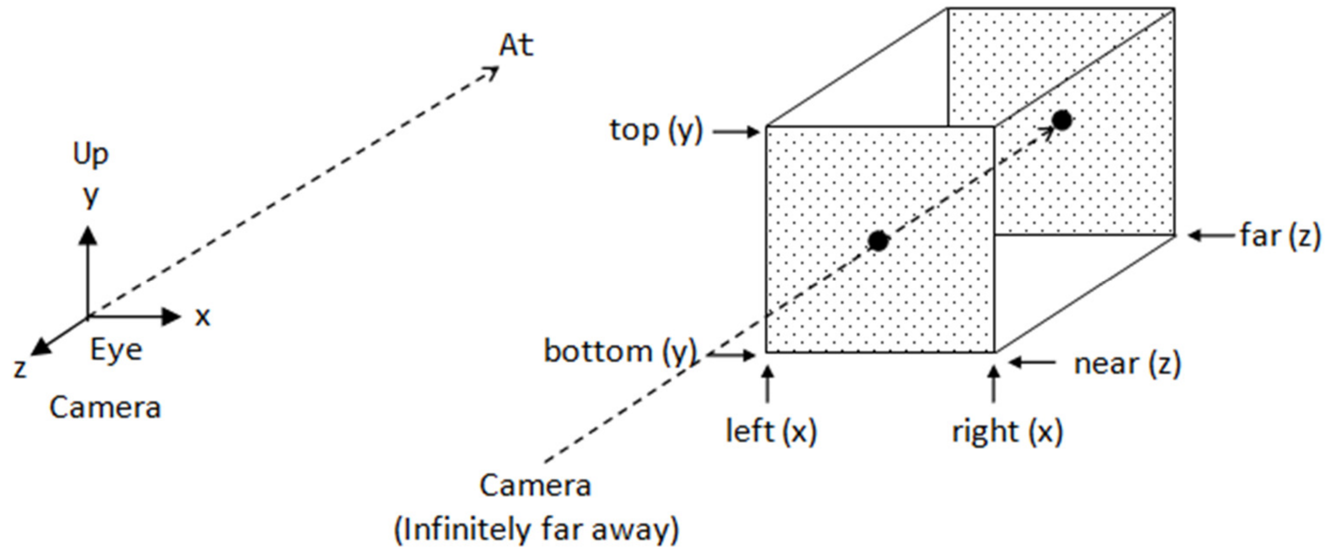
Canonical view volume

Viewport transformation

Image space
(pixel coordinates)

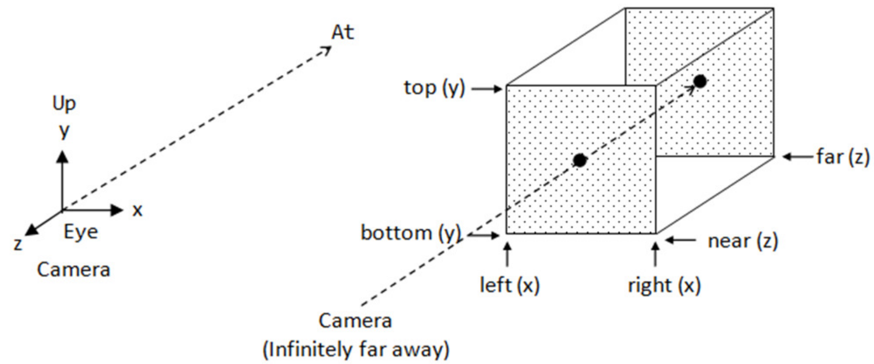


Orthographic View Volume



- Specified by 6 parameters:
  - Right, left, top, bottom, near, far
- Or, if symmetrical:
  - Width, height, near, far

Orthographic Projection Matrix



$$\mathbf{P}_{ortho}(right, left, top, bottom, near, far) = \begin{bmatrix} \frac{2}{right-left} & 0 & 0 & -\frac{right+left}{right-left} \\ 0 & \frac{2}{top-bottom} & 0 & -\frac{top+bottom}{top-bottom} \\ 0 & 0 & \frac{2}{far-near} & \frac{far+near}{far-near} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

In OpenGL:
glOrtho(left, right, bottom, top, near, far)

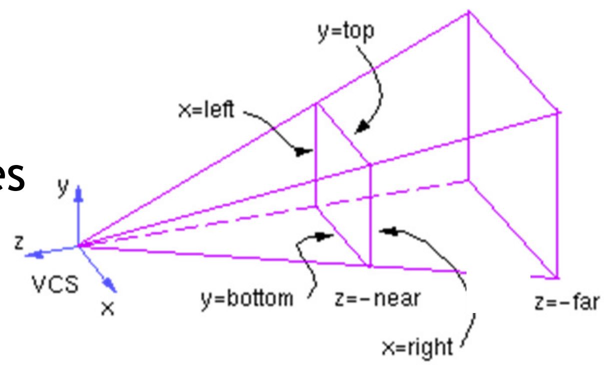
$$\mathbf{P}_{ortho}(width, height, near, far) = \begin{bmatrix} \frac{2}{width} & 0 & 0 & 0 \\ 0 & \frac{2}{height} & 0 & 0 \\ 0 & 0 & \frac{2}{far-near} & \frac{far+near}{far-near} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

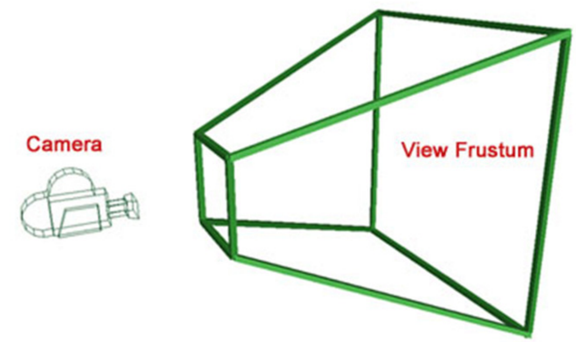
No equivalent in OpenGL

Perspective View Volume

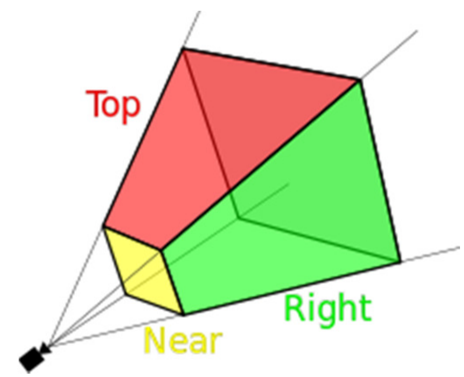
General view volume

Camera coordinates



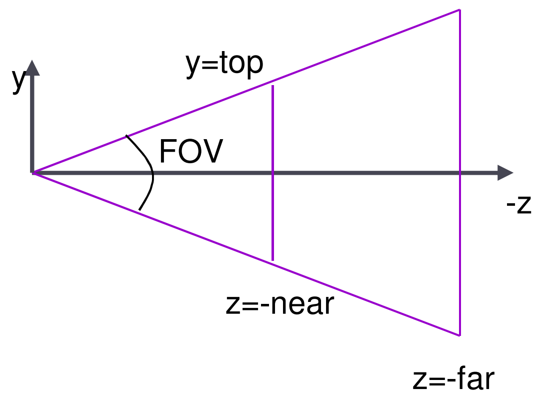


- Defined by 6 parameters, in camera coordinates
  - Left, right, top, bottom boundaries
  - Near, far clipping planes
- Clipping planes to avoid numerical problems
  - Divide by zero
  - Low precision for distant objects
- Usually symmetric, i.e., left=-right, top=-bottom



Perspective View Volume

Symmetrical view volume



- Only 4 parameters
  - Vertical field of view (FOV)
  - Image aspect ratio (width/height)
  - Near, far clipping planes

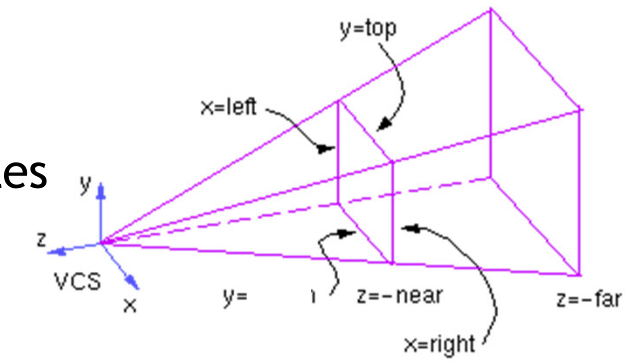
$$\text{aspect ratio}=\frac{right-left}{top-bottom}=\frac{right}{top}$$

$$\tan(FOV/2)=\frac{top}{near}$$

Perspective Projection Matrix

- General view frustum with 6 parameters

Camera coordinates



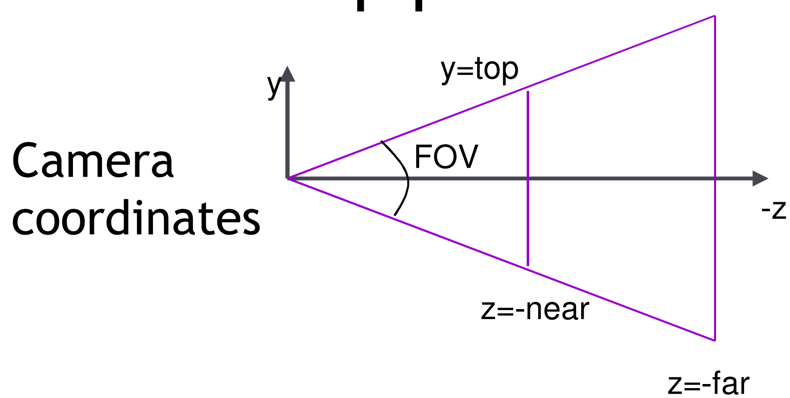
$$\mathbf{P}_{persp}(left, right, top, bottom, near, far) =$$

$$\begin{bmatrix} \frac{2near}{right-left} & 0 & \frac{right+left}{right-left} & 0 \\ 0 & \frac{2near}{top-bottom} & \frac{top+bottom}{top-bottom} & 0 \\ 0 & 0 & \frac{-(far+near)}{far-near} & \frac{-2far \cdot near}{far-near} \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

In OpenGL:
glFrustum(left, right, bottom, top, near, far)

Perspective Projection Matrix

- Symmetrical view frustum with field of view, aspect ratio, near and far clip planes



$$\mathbf{P}_{persp}(FOV, aspect, near, far) = \begin{bmatrix} \frac{1}{aspect \cdot \tan(FOV/2)} & 0 & 0 & 0 \\ 0 & \frac{1}{\tan(FOV/2)} & 0 & 0 \\ 0 & 0 & \frac{near+far}{near-far} & \frac{2 \cdot near \cdot far}{near-far} \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

In OpenGL:
gluPerspective(fov, aspect, near, far)

Canonical View Volume

- Goal: create projection matrix so that
  - User defined view volume is transformed into canonical view volume: cube [-1,1]x[-1,1]x[-1,1]
  - Multiplying corner vertices of view volume by projection matrix and performing homogeneous divide yields corners of canonical view volume
- Perspective and orthographic projection are treated the same way
- Canonical view volume is last stage in which coordinates are in 3D
  - Next step is projection to 2D frame buffer

Viewport Transformation

- After applying projection matrix, scene points are in *normalized viewing coordinates*
  - Per definition within range [-1..1] x [-1..1] x [-1..1]
- Next is projection from 3D to 2D (not reversible)
- Normalized viewing coordinates can be mapped to image (=pixel=frame buffer) coordinates
  - Range depends on window (view port) size: [x0…x1] x [y0…y1]
- Scale and translation required:

$$\mathbf{D}(x_0, x_1, y_0, y_1) = \begin{bmatrix} (x_1 - x_0)/2 & 0 & 0 & (x_0 + x_1)/2 \\ 0 & (y_1 - y_0)/2 & 0 & (y_0 + y_1)/2 \\ 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Lecture Overview

- View Volumes
- Vertex Transformation
- Rendering Pipeline
- Culling

Complete Vertex Transformation

- Mapping a 3D point in object coordinates to pixel coordinates:

$$\mathbf{p}' = \mathbf{DPC}^{-1}\mathbf{M}\mathbf{p}$$

Object space

- **M**: Object-to-world matrix
- **C**: camera matrix
- **P**: projection matrix
- **D**: viewport matrix

Complete Vertex Transformation

- Mapping a 3D point in object coordinates to pixel coordinates:

$$\mathbf{p}' = \mathbf{D}\mathbf{P}\mathbf{C}^{-1}\mathbf{M}\mathbf{p}$$

Object space

World space

  - **M**: Object-to-world matrix
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Object space

World space

Camera space

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Complete Vertex Transformation

- Mapping a 3D point in object coordinates to pixel coordinates:

$$\mathbf{p}' = \mathbf{D}\mathbf{P}\mathbf{C}^{-1}\mathbf{M}\mathbf{p}$$

- Object space
- World space
- Camera space
- Canonical view volume

  - **M**: Object-to-world matrix
  - **C**: camera matrix
  - **P**: projection matrix
  - **D**: viewport matrix

Complete Vertex Transformation

- Mapping a 3D point in object coordinates to pixel coordinates: $\mathbf{p}' = \mathbf{D}\mathbf{P}\mathbf{C}^{-1}\mathbf{M}\mathbf{p}$

  Object space

  World space

  Camera space

  Canonical view volume

  Image space

  - **M**: Object-to-world matrix
  - **C**: camera matrix
  - **P**: projection matrix
  - **D**: viewport matrix

Complete Vertex Transformation

- Mapping a 3D point in object coordinates to pixel coordinates:

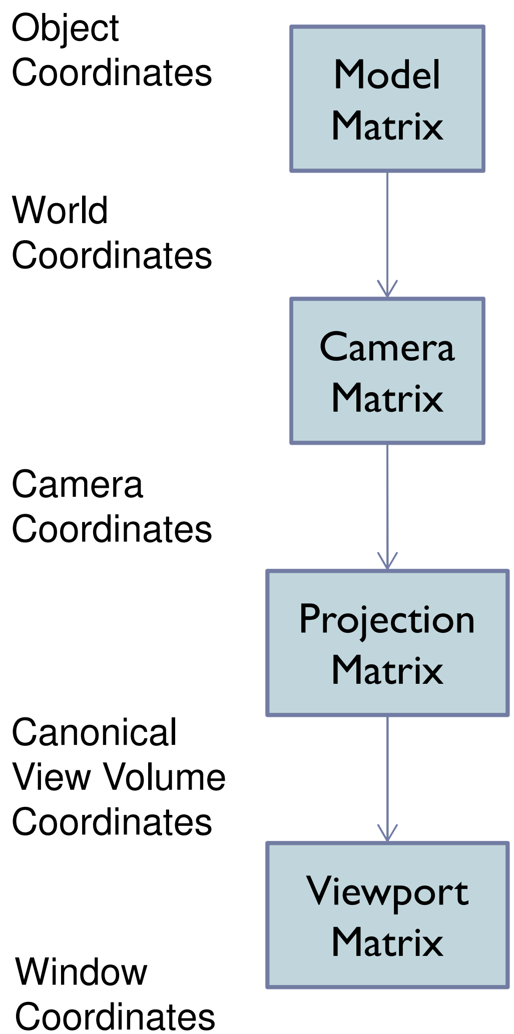
$$\mathbf{p}' = \mathbf{D}\mathbf{P}\mathbf{C}^{-1}\mathbf{M}\mathbf{p}$$

$$\mathbf{p}' = \begin{bmatrix} x' \\ y' \\ z' \\ w' \end{bmatrix}$$

Pixel coordinates: x'/w' y'/w'

  - **M**: Object-to-world matrix
  - **C**: camera matrix
  - **P**: projection matrix
  - **D**: viewport matrix

The Complete Vertex Transformation



Complete Vertex Transformation in OpenGL

- Mapping a 3D point in object coordinates to pixel coordinates:

OpenGL GL_MODELVIEW matrix

$$\mathbf{p}' = \mathbf{D}\mathbf{P}\mathbf{C}^{-1}\mathbf{M}\mathbf{p}$$

OpenGL GL_PROJECTION matrix

  - **M**: Object-to-world matrix
  - **C**: camera matrix
  - **P**: projection matrix
  - **D**: viewport matrix

Complete Vertex Transformation in OpenGL

- GL_MODELVIEW, $\mathbf{C^{-1}M}$
  - Defined by the programmer.
  - Think of the ModelView matrix as where you stand with the camera and the direction you point it.
- GL_PROJECTION, **P**
  - Utility routines to set it by specifying view volume: glFrustum(), gluPerspective(), glOrtho()
  - Think of the projection matrix as describing the attributes of your camera, such as field of view, focal length, etc.
- Viewport, **D**
  - Specify implicitly via glViewport()
  - No direct access with equivalent to GL_MODELVIEW or GL_PROJECTION