CSE 167:
Introduction to Computer Graphics
Lecture #11: Polynomial Curves

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Announcements

- Project 3 due this Friday
- Midterm to be returned this Thursday

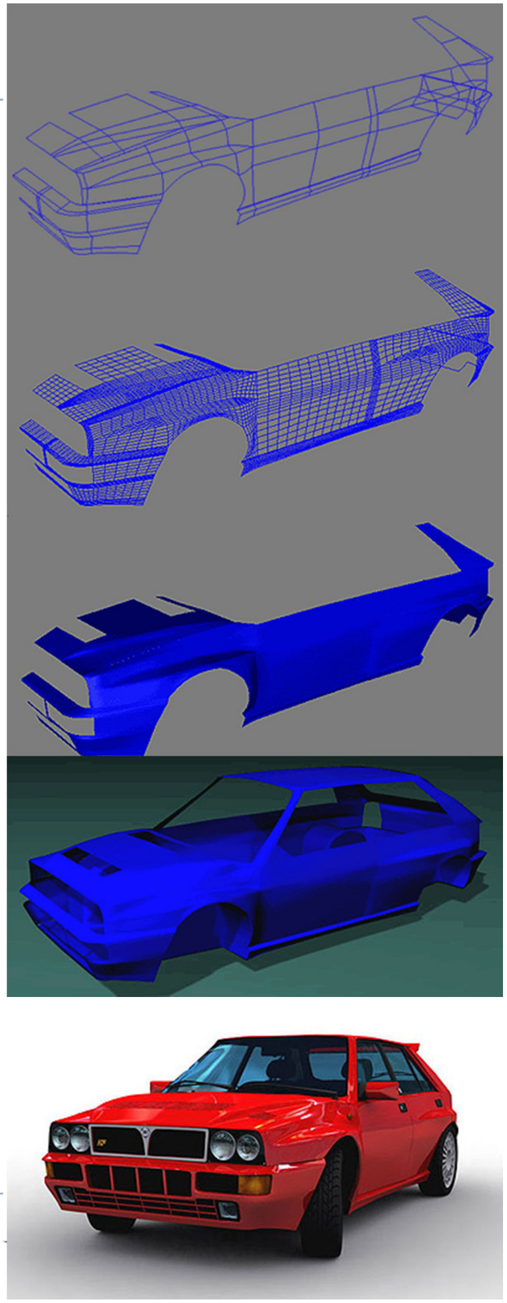
Lecture Overview

- Polynomial Curves
  - Introduction
  - Polynomial functions
- Bézier Curves
  - Introduction
  - Drawing Bézier curves
  - Piecewise Bézier curves

Modeling

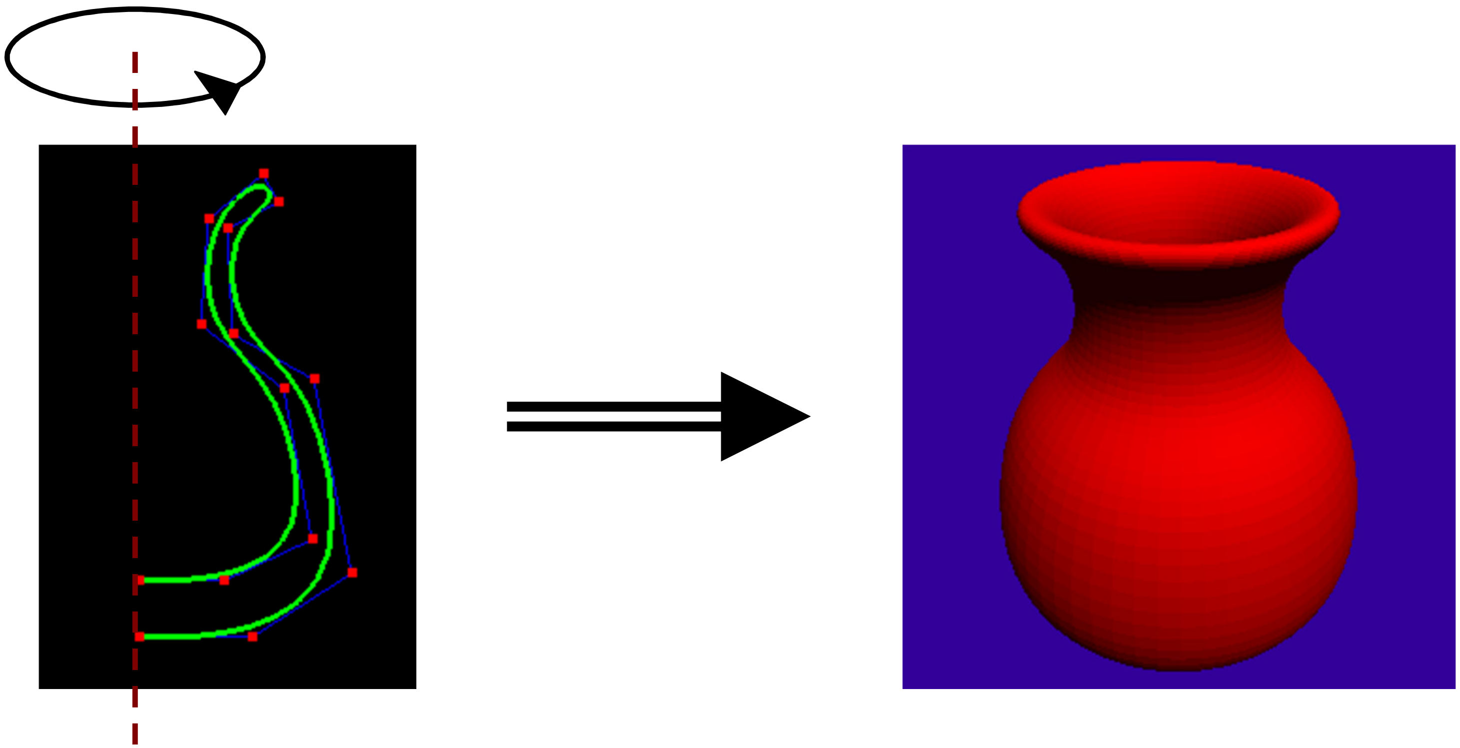
- Creating 3D objects
- How to construct complex surfaces?
- Goal
  - Specify objects with control points
  - Objects should be visually pleasing (smooth)
- Start with curves, then generalize to surfaces

- Next: What can curves be used for?



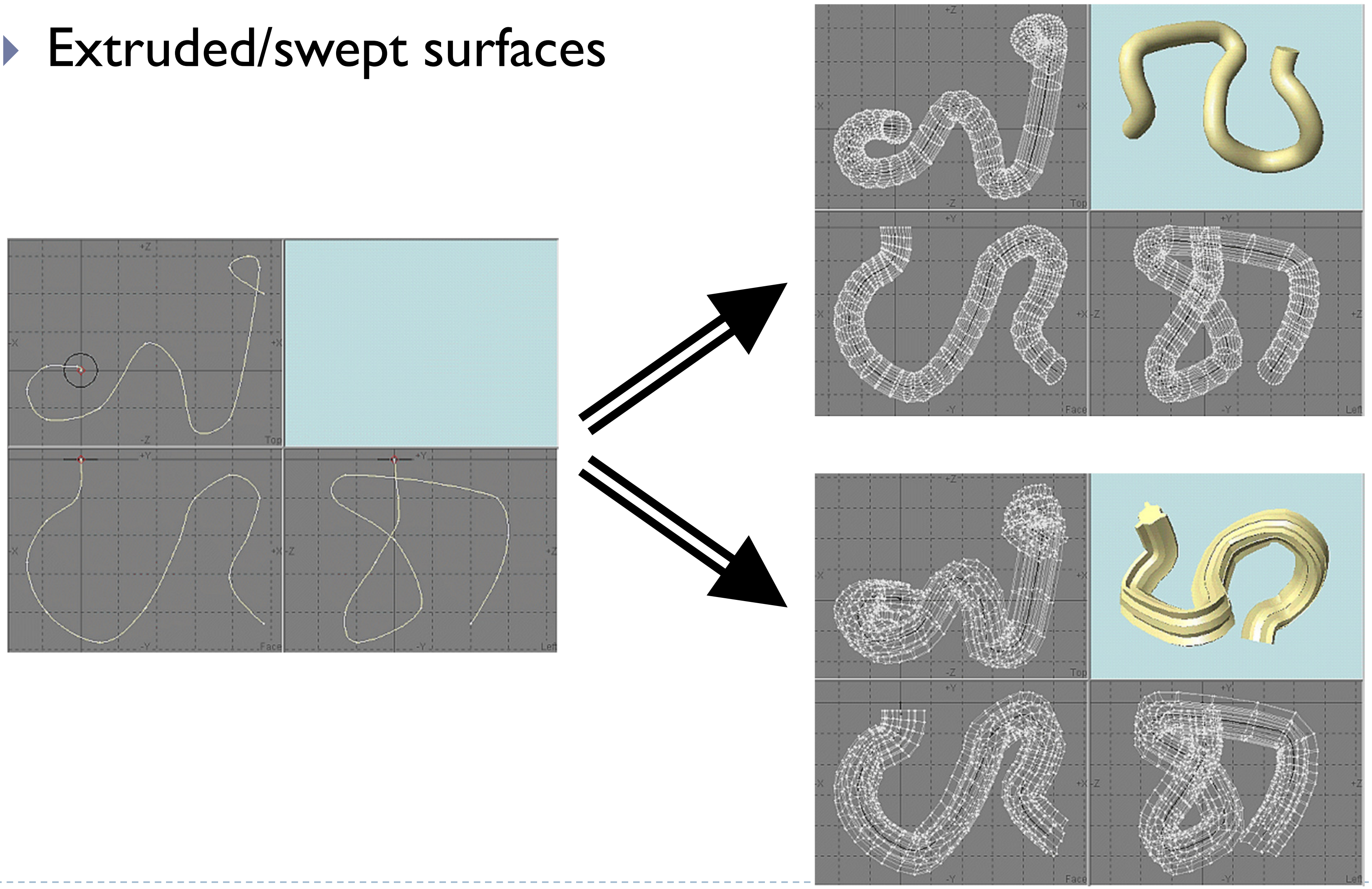
Curves

- Surface of revolution



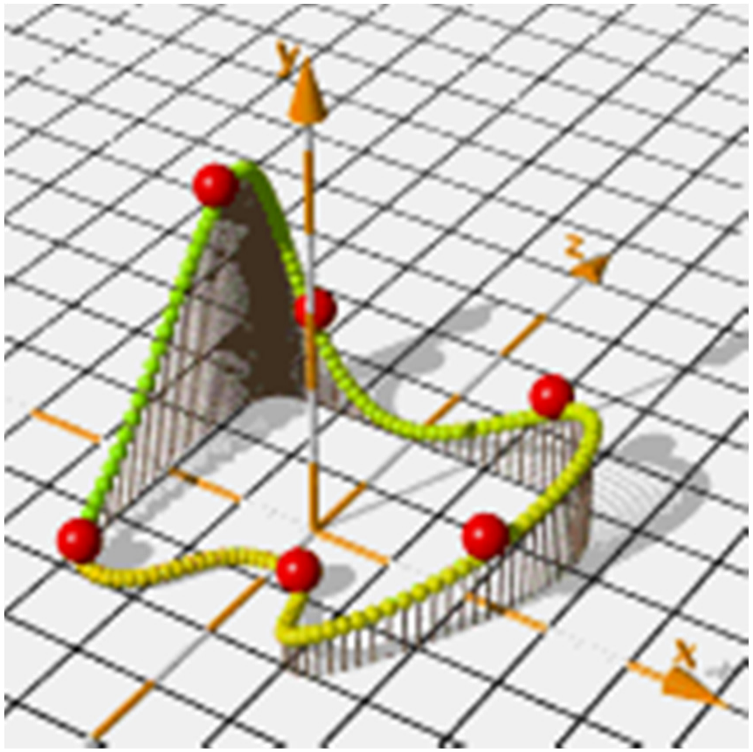
Curves

- Extruded/swept surfaces



Curves

- Animation
  - Provide a "track" for objects
  - Use as camera path



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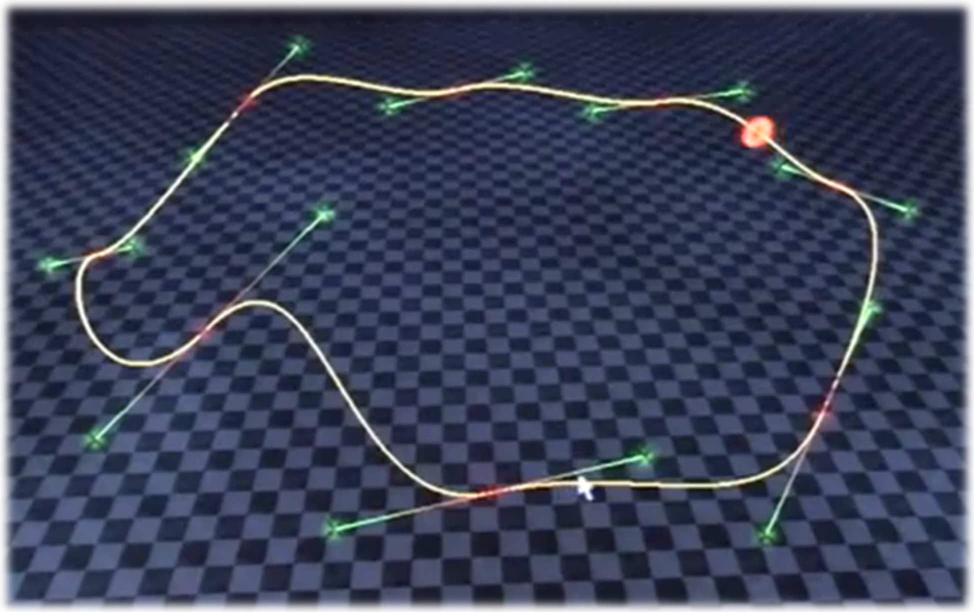
Video

- Bezier Curves
  - http://www.youtube.com/watch?v=hlDYJNEiYvU



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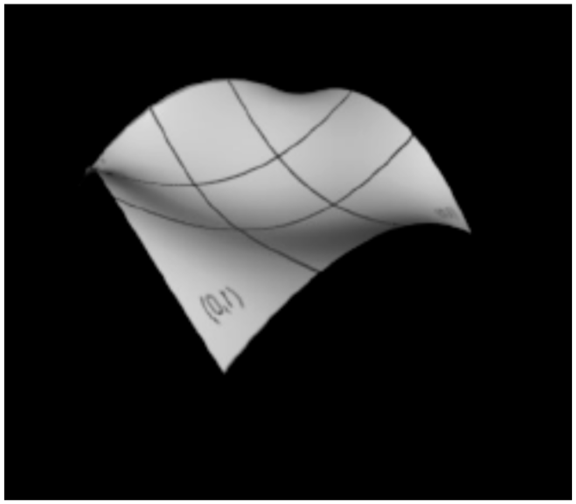
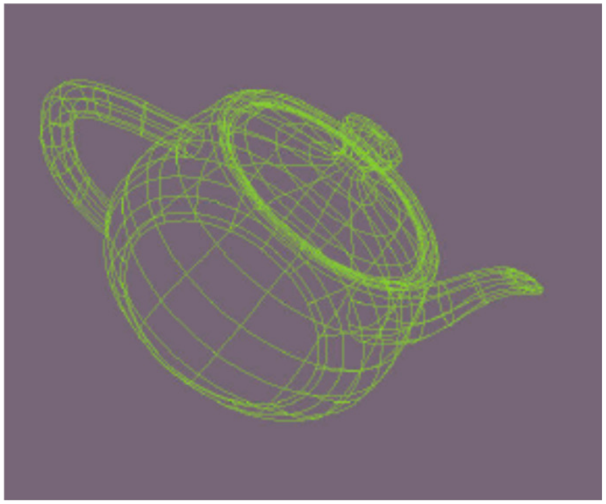
Curves

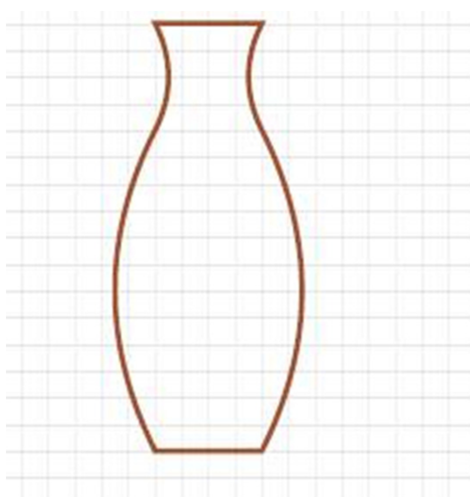
- Can be generalized to surface patches

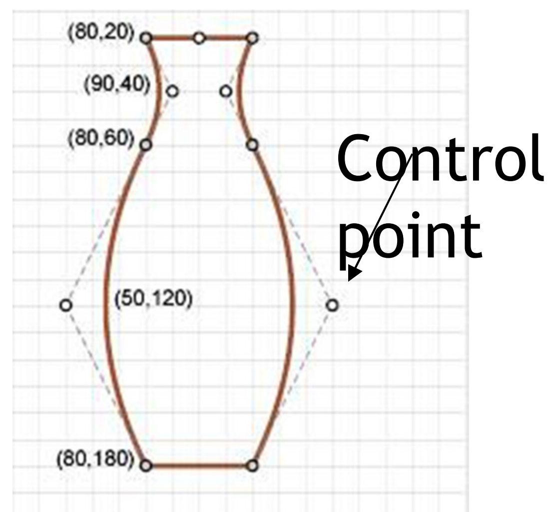




Curve Representation

- Specify many points along a curve, connect with lines?
  - Difficult to get precise, smooth results across magnification levels
  - Large storage and CPU requirements
  - How many points are enough?
- Specify a curve using a small number of "control points"
  - Known as a *spline curve* or just *spline*





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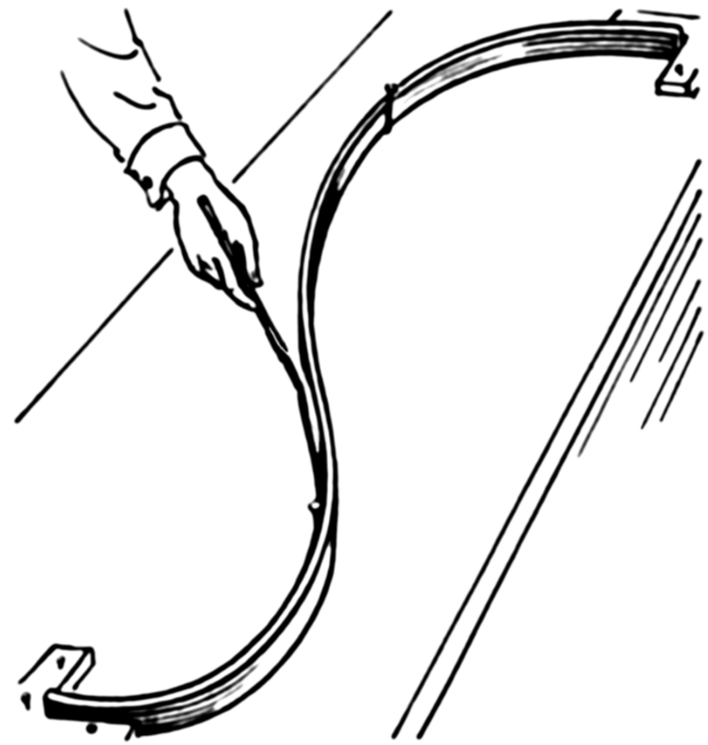
Spline: Definition

- Wikipedia:
  - Term comes from flexible spline devices used by shipbuilders and draftsmen to draw smooth shapes.
  - Spline consists of a long strip fixed in position at a number of points that relaxes to form a smooth curve passing through those points.



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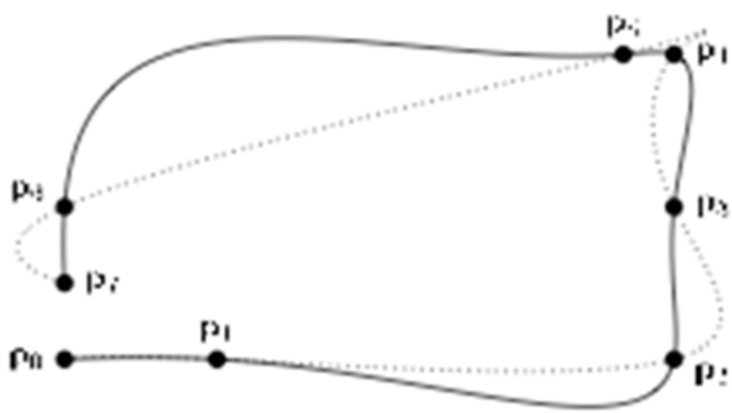
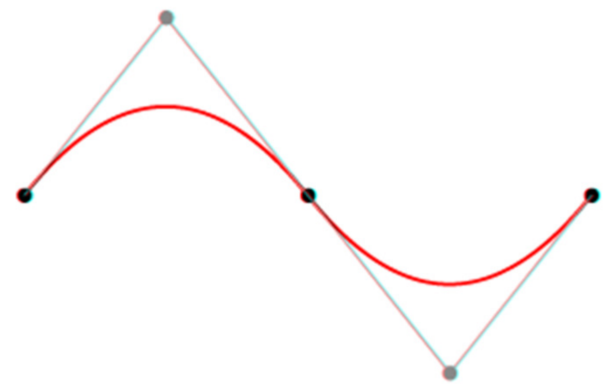
Interpolating Control Points

- “Interpolating” means that curve goes through all control points
- Seems most intuitive
- Surprisingly, not usually the best choice
  - Hard to predict behavior
  - Hard to get aesthetically pleasing curves



Approximating Control Points

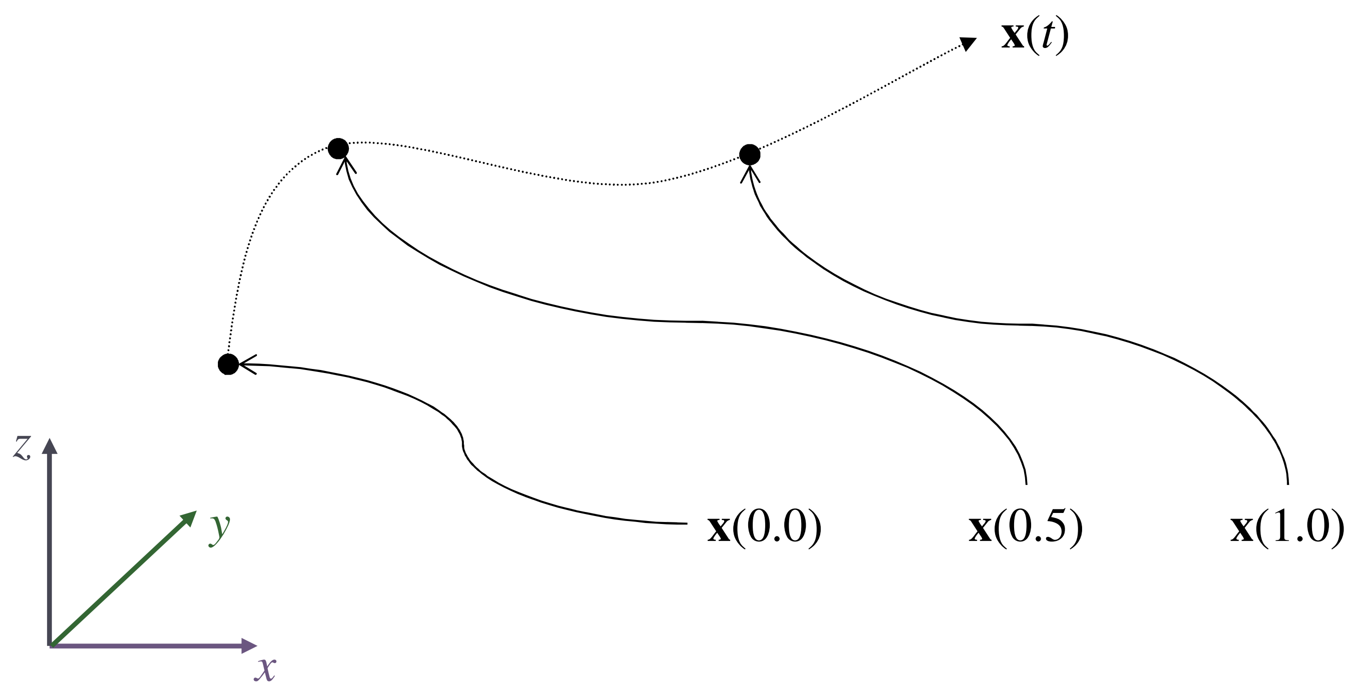
- Curve is "influenced" by control points



- Various types
- Most common: polynomial functions
  - Bézier spline (our focus)
  - B-spline (generalization of Bézier spline)
  - NURBS (Non Uniform Rational Basis Spline): used in CAD tools

Mathematical Definition

- A vector valued function of one variable $\mathbf{x}(t)$
  - Given t, compute a 3D point $\mathbf{x}=(x,y,z)$
  - Could be interpreted as three functions: $x(t)$, $y(t)$, $z(t)$
  - Parameter t "moves a point along the curve"



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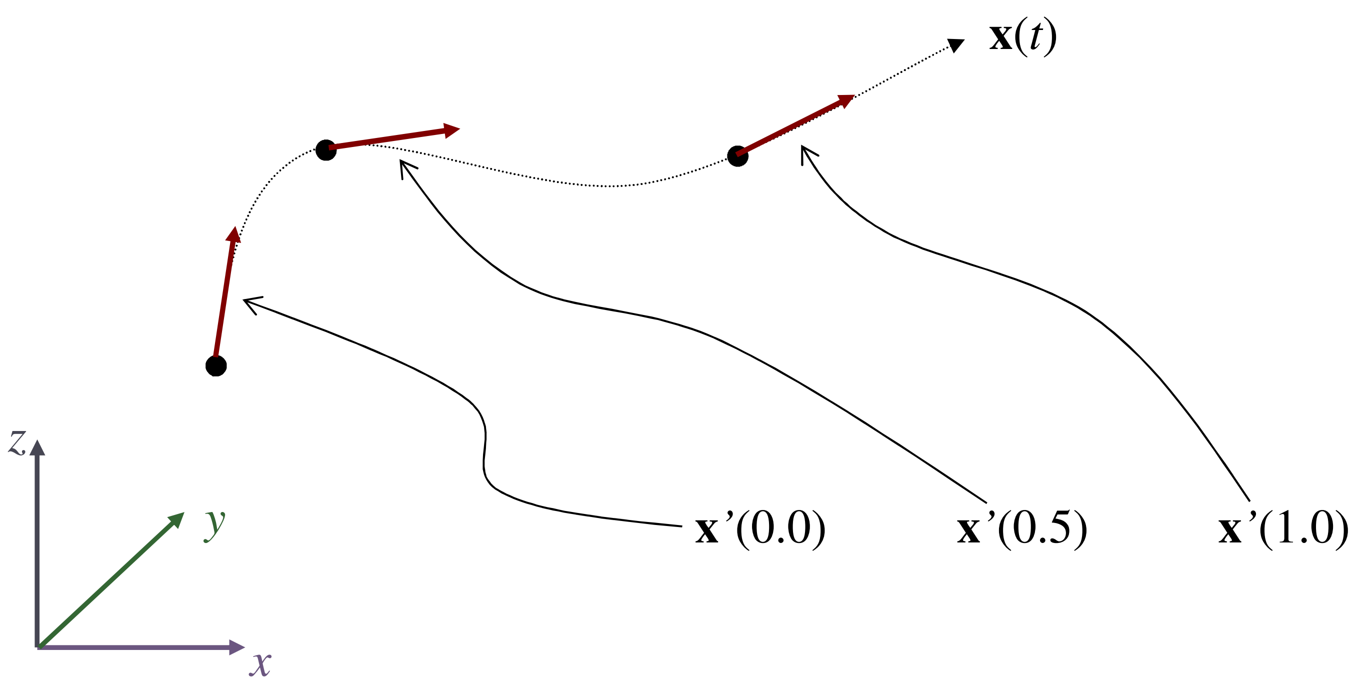
Tangent Vector

- Derivative $\mathbf{x}'(t) = \frac{d\mathbf{x}}{dt} = (x'(t), y'(t), z'(t))$
- Vector x' points in direction of movement
- Length corresponds to speed



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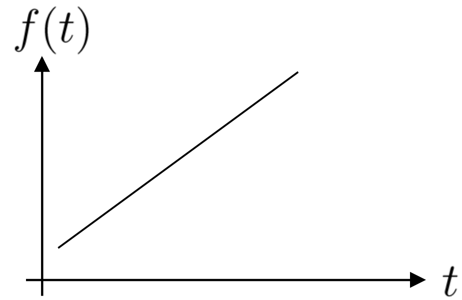
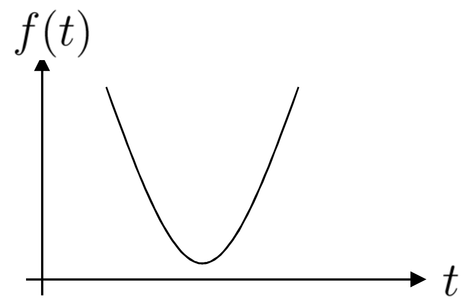
Polynomial Functions

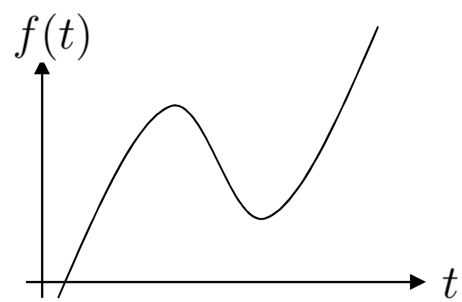
- Linear: (1st order) $$f(t) = at + b$$



- Quadratic: (2nd order) $$f(t) = at^2 + bt + c$$



- Cubic: (3rd order) $$f(t) = at^3 + bt^2 + ct + d$$



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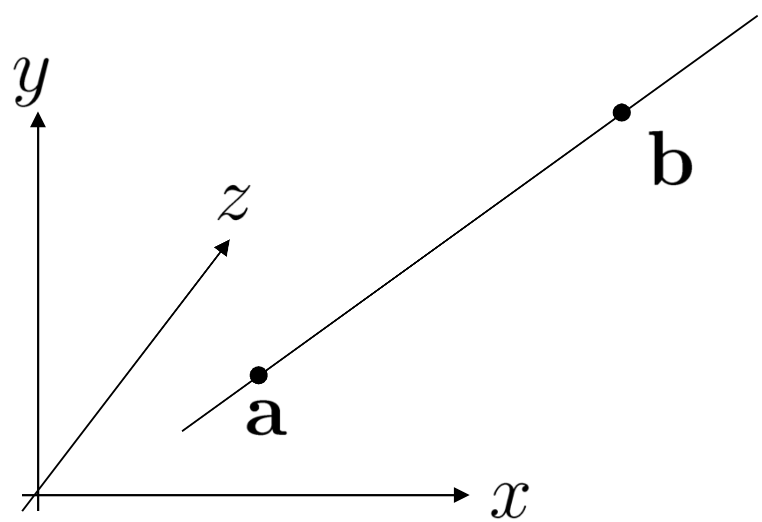
Polynomial Curves

- Linear $\mathbf{x}(t) = \mathbf{a}t + \mathbf{b}$

$$\mathbf{x} = (x, y, z), \mathbf{a} = (a_x, a_y, a_z), \mathbf{b} = (b_x, b_y, b_z)$$

- Evaluated as:

$$x(t) = a_x t + b_x$$
$$y(t) = a_y t + b_y$$
$$z(t) = a_z t + b_z$$



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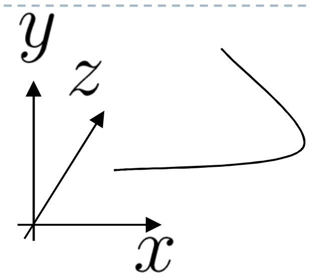
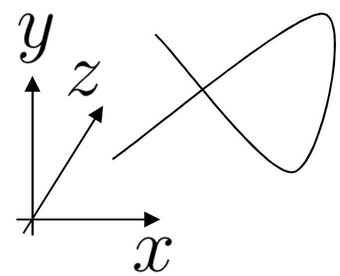
Polynomial Curves

- Quadratic: $\mathbf{x}(t) = \mathbf{a}t^2 + \mathbf{b}t + \mathbf{c}$
  (2nd order)



- Cubic: $\mathbf{x}(t) = \mathbf{a}t^3 + \mathbf{b}t^2 + \mathbf{c}t + \mathbf{d}$
  (3rd order)



- We usually define the curve for $0 \leq t \leq 1$

Control Points

- Polynomial coefficients **a, b, c, d** can be interpreted as *control points*
  - Remember: **a, b, c, d** have *x,y,z* components each
- Unfortunately, they do not intuitively describe the shape of the curve
- Goal: intuitive control points

Control Points

- How many control points?
  - Two points define a line (1st order)
  - Three points define a quadratic curve (2nd order)
  - Four points define a cubic curve (3rd order)
  - $k+1$ points define a k-order curve
- Let's start with a line…

First Order Curve

- Based on linear interpolation (LERP)
  - Weighted average between two values
  - "Value" could be a number, vector, color, …
- Interpolate between points $\mathbf{p}_0$ and $\mathbf{p}_1$ with parameter t
  - Defines a "curve" that is straight (first-order spline)
  - $t=0$ corresponds to $\mathbf{p}_0$
  - $t=1$ corresponds to $\mathbf{p}_1$
  - $t=0.5$ corresponds to midpoint

$$\mathbf{x}(t) = Lerp\left(t,\ \mathbf{p}_0,\ \mathbf{p}_1\right) = \left(1-t\right)\mathbf{p}_0 + t\ \mathbf{p}_1$$

Linear Interpolation

- Three equivalent ways to write it
  - Expose different properties

1. Regroup for points **p**

$$\mathbf{x}(t) = \mathbf{p}_0(1-t) + \mathbf{p}_1 t$$

2. Regroup for t

$$\mathbf{x}(t) = (\mathbf{p}_1 - \mathbf{p}_0)t + \mathbf{p}_0$$

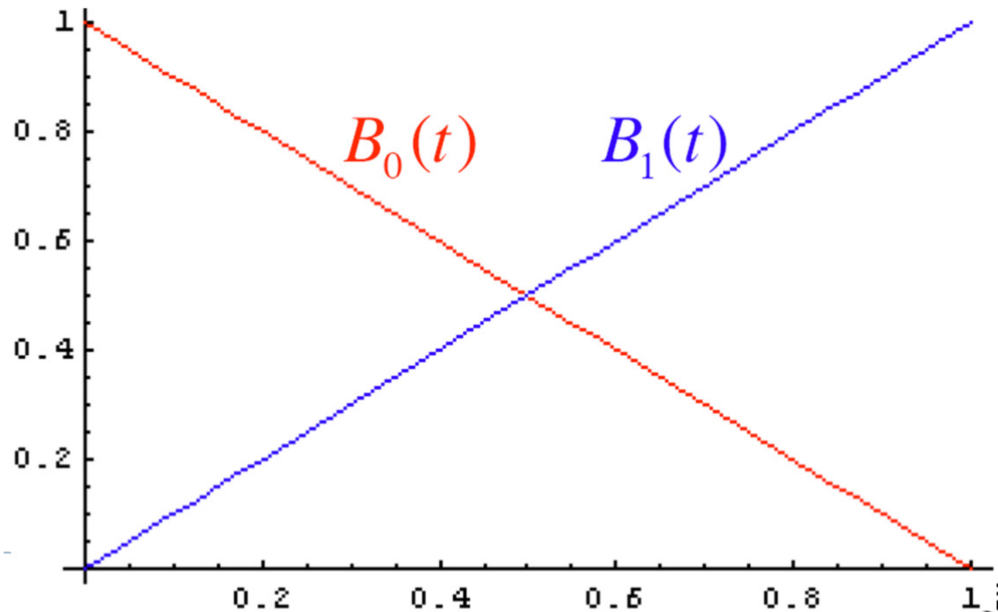
3. Matrix form

$$\mathbf{x}(t) = \begin{bmatrix} \mathbf{p}_0 & \mathbf{p}_1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} t \\ 1 \end{bmatrix}$$

Weighted Average

$$\mathbf{x}(t) = (1-t)\mathbf{p}_0 + \quad (t)\mathbf{p}_1$$
$$= B_0(t)\ \mathbf{p}_0 + B_1(t)\mathbf{p}_1, \text{ where } B_0(t) = 1-t \text{ and } B_1(t) = t$$

- Weights are a function of t
  - Sum is always 1, for any value of t
  - Also known as *blending functions*



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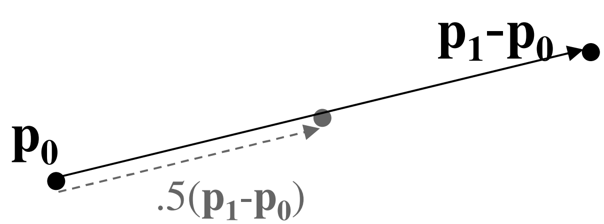
Linear Polynomial

$$\mathbf{x}(t) = \underbrace{(\mathbf{p}_1 - \mathbf{p}_0)}_{\substack{\text{vector} \\ \mathbf{a}}}\, t + \underbrace{\mathbf{p}_0}_{\substack{\text{point} \\ \mathbf{b}}}$$

- Curve is based at point $\mathbf{p_0}$
- Add the vector, scaled by t



Matrix Form

$$\mathbf{x}(t) = \begin{bmatrix} \mathbf{p}_0 & \mathbf{p}_1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} t \\ 1 \end{bmatrix} = \mathbf{GBT}$$

- Geometry matrix $\mathbf{G} = \begin{bmatrix} \mathbf{p}_0 & \mathbf{p}_1 \end{bmatrix}$
- Geometric basis $\mathbf{B} = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$
- Polynomial basis $T = \begin{bmatrix} t \\ 1 \end{bmatrix}$
- In components

$$\mathbf{x}(t) = \begin{bmatrix} p_{0x} & p_{1x} \\ p_{0y} & p_{1y} \\ p_{0z} & p_{1z} \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} t \\ 1 \end{bmatrix}$$

Matrix Form

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Tangent

- For a straight line, the tangent is constant

$$\mathbf{x}'(t) = \mathbf{p}_1 - \mathbf{p}_0$$

- Weighted average $\mathbf{x}'(t) = (-1)\mathbf{p}_0 + (+1)\mathbf{p}_1$

- Polynomial $\mathbf{x}'(t) = 0t + (\mathbf{p}_1 - \mathbf{p}_0)$

- Matrix form $\mathbf{x}'(t) = \begin{bmatrix} \mathbf{p}_0 & \mathbf{p}_1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$