CSE 167: Introduction to Computer Graphics Lecture #4: Coordinate Systems

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Announcements

- Tomorrow: Discussion at 3pm in Center Hall 109
- Next Friday: homework 2 due at 2pm
  - Upload to TritonEd
  - Demonstrate in CSE basement labs

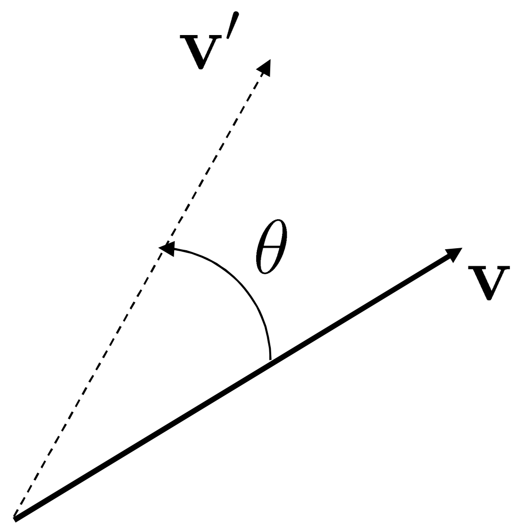
Today

- Finish up linear algebra foundations
- Coordinate system transformations

Rotation in 2D

- Convention: positive angle rotates counterclockwise
- Rotation matrix

$$\mathbf{R}(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$



$$\mathbf{v}' = \mathbf{R}(\theta)\mathbf{v}$$

Rotation in 3D

Rotation around coordinate axes

$$\mathbf{R}_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

$$\mathbf{R}_y(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$\mathbf{R}_z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Rotation in 3D

- Concatenation of rotations around x, y, z axes

$$\mathbf{R}_{x,y,z}(\theta_x, \theta_y, \theta_z) = \mathbf{R}_x(\theta_x)\mathbf{R}_y(\theta_y)\mathbf{R}_z(\theta_z)$$

- $\theta_x, \theta_y, \theta_z$ are called Euler angles
- Result depends on matrix order!

$$\mathbf{R}_x(\theta_x)\mathbf{R}_y(\theta_y)\mathbf{R}_z(\theta_z) \neq \mathbf{R}_z(\theta_z)\mathbf{R}_y(\theta_y)\mathbf{R}_x(\theta_x)$$

Rotation about an Arbitrary Axis

- Complicated!
- Rotate point [x,y,z] about axis [u,v,w] by angle θ:

$$\begin{bmatrix} \dfrac{u(ux+vy+wz)(1-\cos\theta)+(u^2+v^2+w^2)x\cos\theta+\sqrt{u^2+v^2+w^2}(-wy+vz)\sin\theta}{u^2+v^2+w^2} \\ \dfrac{v(ux+vy+wz)(1-\cos\theta)+(u^2+v^2+w^2)y\cos\theta+\sqrt{u^2+v^2+w^2}(wx-uz)\sin\theta}{u^2+v^2+w^2} \\ \dfrac{w(ux+vy+wz)(1-\cos\theta)+(u^2+v^2+w^2)z\cos\theta+\sqrt{u^2+v^2+w^2}(-vx+uy)\sin\theta}{u^2+v^2+w^2} \end{bmatrix}$$

Concatenating transformations

- Given a sequence of transformations $\mathbf{M}_3\mathbf{M}_2\mathbf{M}_1$

$$\mathbf{p}' = \mathbf{M}_3\mathbf{M}_2\mathbf{M}_1\mathbf{p}$$

$$\mathbf{M}_{total} = \mathbf{M}_3\mathbf{M}_2\mathbf{M}_1$$

$$\mathbf{p}' = \mathbf{M}_{total}\mathbf{p}$$

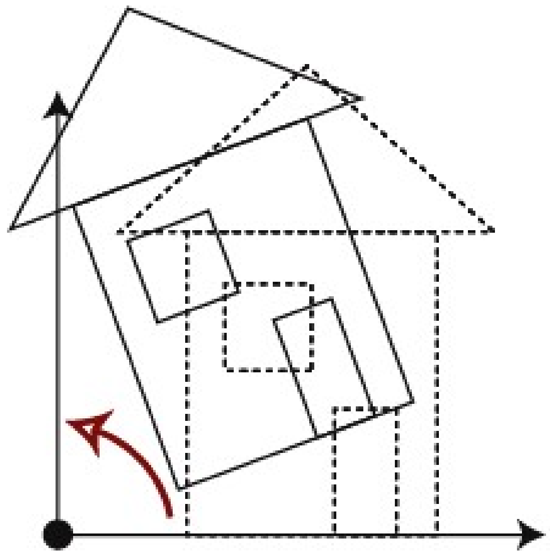
- Note: associativity applies

$$\mathbf{M}_{total} = (\mathbf{M}_3\mathbf{M}_2)\mathbf{M}_1 = \mathbf{M}_3(\mathbf{M}_2\mathbf{M}_1)$$

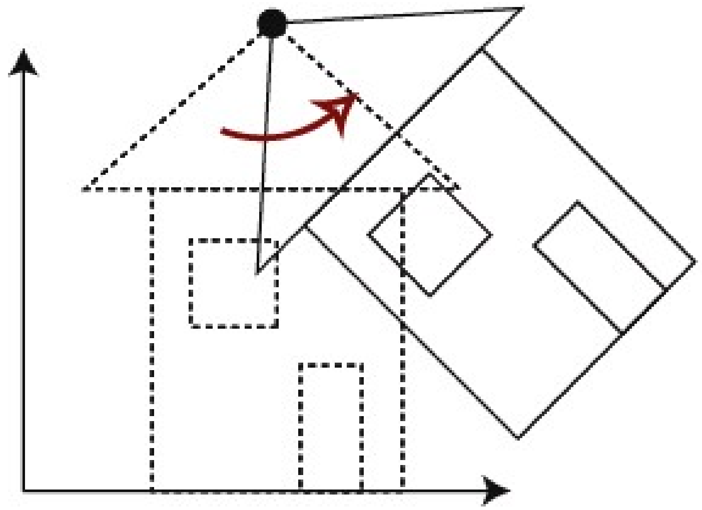
- Efficient inversion (when the components are "simple")

$$\mathbf{M}_{total}^{-1} = \mathbf{M}_1^{-1}\mathbf{M}_2^{-1}\mathbf{M}_3^{-1}$$

How to rotate around a Pivot Point?

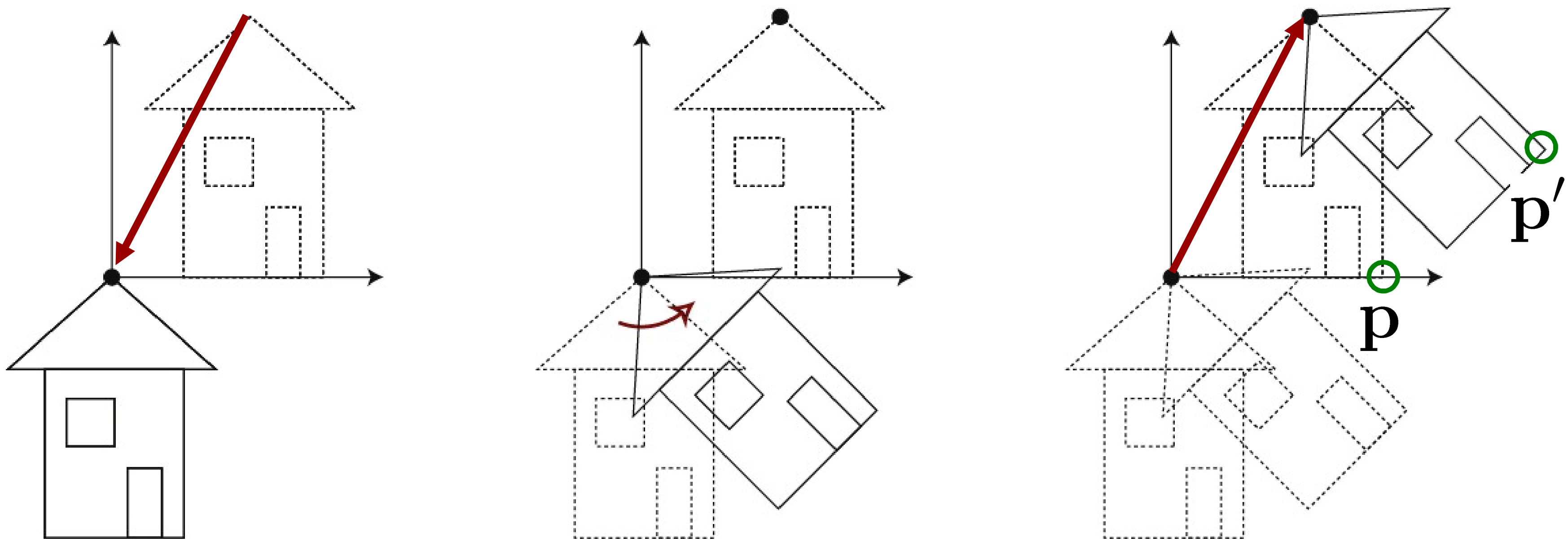


Rotation around origin:
p' = R p



Rotation around pivot point:
p' = ?

Rotating point p around a pivot point



1. Translation T 2. Rotation R 3. Translation T^{-1}

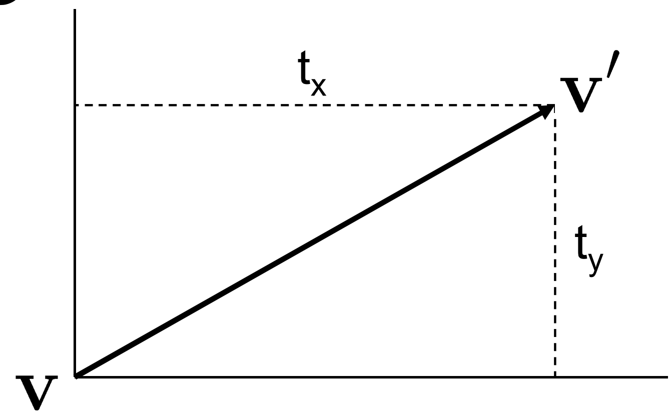
$$p' = T^{-1} R\, T\, p$$

Today

- Vectors and matrices
- Affine transformations
- Homogeneous coordinates

Translation

- Translation in 2D



- Translation matrix T=?

$$v' = \begin{bmatrix} v_x \\ v_y \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \end{bmatrix} = Tv = T\begin{bmatrix} v_x \\ v_y \end{bmatrix} = \begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix}\begin{bmatrix} v_x \\ v_y \end{bmatrix}$$

Translation

- Translation in 2D: 3x3 matrix

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

- Analogous in 3D: 4x4 matrix

$$\begin{bmatrix} x' \\ y' \\ z' \\ w \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

Homogeneous Coordinates

- A trick to unify and simplify computations, in particular:
  - affine transformations (esp. rotation, scaling, translation)
  - projective transformations

Homogeneous Coordinates

- Add an extra component. 1 for a point, 0 for a vector:

$$\mathbf{p} = \begin{bmatrix} p_x \\ p_y \\ p_z \\ 1 \end{bmatrix} \qquad \vec{\mathbf{v}} = \begin{bmatrix} v_x \\ v_y \\ v_z \\ 0 \end{bmatrix}$$

- Combine **M** and **d** into single 4x4 matrix:

$$\begin{bmatrix} m_{xx} & m_{xy} & m_{xz} & d_x \\ m_{yx} & m_{yy} & m_{yz} & d_y \\ m_{zx} & m_{zy} & m_{zz} & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- And see what happens when we multiply…

Homogeneous Point Transform

- Transform a point:

$$\begin{bmatrix} p'_x \\ p'_y \\ p'_z \\ 1 \end{bmatrix} = \begin{bmatrix} m_{xx} & m_{xy} & m_{xz} & d_x \\ m_{yx} & m_{yy} & m_{yz} & d_y \\ m_{zx} & m_{zy} & m_{zz} & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ p_z \\ 1 \end{bmatrix} = \begin{bmatrix} m_{xx}p_x + m_{xy}p_y + m_{xz}p_z + d_x \\ m_{yx}p_x + m_{yy}p_y + m_{yz}p_z + d_y \\ m_{zx}p_x + m_{zy}p_y + m_{zz}p_z + d_z \\ 0+0+0+1 \end{bmatrix}$$

$$\mathbf{M} \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} + \vec{\mathbf{d}}$$

- Top three rows are the affine transform!
- Bottom row stays 1

Homogeneous Vector Transform

- Transform a vector:

$$\begin{bmatrix} v'_x \\ v'_y \\ v'_z \\ 0 \end{bmatrix} = \begin{bmatrix} m_{xx} & m_{xy} & m_{xz} & d_x \\ m_{yx} & m_{yy} & m_{yz} & d_y \\ m_{zx} & m_{zy} & m_{zz} & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_x \\ v_y \\ v_z \\ 0 \end{bmatrix} = \begin{bmatrix} m_{xx}v_x + m_{xy}v_y + m_{xz}v_z + 0 \\ m_{yx}v_x + m_{yy}v_y + m_{yz}v_z + 0 \\ m_{zx}v_x + m_{zy}v_y + m_{zz}v_z + 0 \\ 0+0+0+0 \end{bmatrix}$$

$$\mathbf{M}\begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}$$

- Top three rows are the linear transform
  - Displacement **d** is properly ignored
- Bottom row stays 0

Homogeneous Arithmetic

- Legal operations always end in 0 or 1!

$$\text{vector+vector:} \quad \begin{bmatrix} \vdots \\ 0 \end{bmatrix} + \begin{bmatrix} \vdots \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} \vdots \\ 0 \end{bmatrix}$$

$$\text{vector-vector:} \quad \begin{bmatrix} \vdots \\ 0 \end{bmatrix} - \begin{bmatrix} \vdots \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} \vdots \\ 0 \end{bmatrix}$$

$$\text{scalar*vector:} \quad s\begin{bmatrix} \vdots \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} \vdots \\ 0 \end{bmatrix}$$

$$\text{point+vector:} \quad \begin{bmatrix} \vdots \\ 1 \end{bmatrix} + \begin{bmatrix} \vdots \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} \vdots \\ 1 \end{bmatrix}$$

$$\text{point-point:} \quad \begin{bmatrix} \vdots \\ 1 \end{bmatrix} - \begin{bmatrix} \vdots \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} \vdots \\ 0 \end{bmatrix}$$

$$\text{point+point:} \quad \begin{bmatrix} \vdots \\ 1 \end{bmatrix} + \begin{bmatrix} \vdots \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} \vdots \\ 2 \end{bmatrix}$$

$$\text{scalar*point:} \quad s\begin{bmatrix} \vdots \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} \vdots \\ s \end{bmatrix}$$

$$\left\{ \begin{matrix} \text{weighted average} \\ \text{affine combination} \end{matrix} \right\} \text{ of points:} \quad \frac{1}{3}\begin{bmatrix} \vdots \\ 1 \end{bmatrix} + \frac{2}{3}\begin{bmatrix} \vdots \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} \vdots \\ 1 \end{bmatrix}$$

Homogeneous Transforms

- Rotation, Scale, and Translation of points and vectors unified in a single matrix transformation:

$$\mathbf{p}' = \mathbf{M}\ \mathbf{p}$$

- Matrix has the form:
  - Last row always 0,0,0,1

$$\begin{bmatrix} m_{xx} & m_{xy} & m_{xz} & d_x \\ m_{yx} & m_{yy} & m_{yz} & d_y \\ m_{zx} & m_{zy} & m_{zz} & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Transforms can be composed by matrix multiplication
  - Same caveat: order of operations is important
  - Same note: transforms operate right-to-left

4x4 Scale Matrix

- Generic form:

$$\begin{bmatrix} s & 0 & 0 & 0 \\ 0 & t & 0 & 0 \\ 0 & 0 & u & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Inverse:

$$\begin{bmatrix} \frac{1}{s} & 0 & 0 & 0 \\ 0 & \frac{1}{t} & 0 & 0 \\ 0 & 0 & \frac{1}{u} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

4x4 Rotation Matrix

- Generic form: $$\begin{bmatrix} r_1 & r_2 & r_3 & 0 \\ r_4 & r_5 & r_6 & 0 \\ r_7 & r_8 & r_9 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Inverse = transpose: $$\begin{bmatrix} r_1 & r_4 & r_7 & 0 \\ r_2 & r_5 & r_8 & 0 \\ r_3 & r_6 & r_9 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
    - Rotation matrices are orthogonal

4x4 Translation Matrix

- Generic form: $$\begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

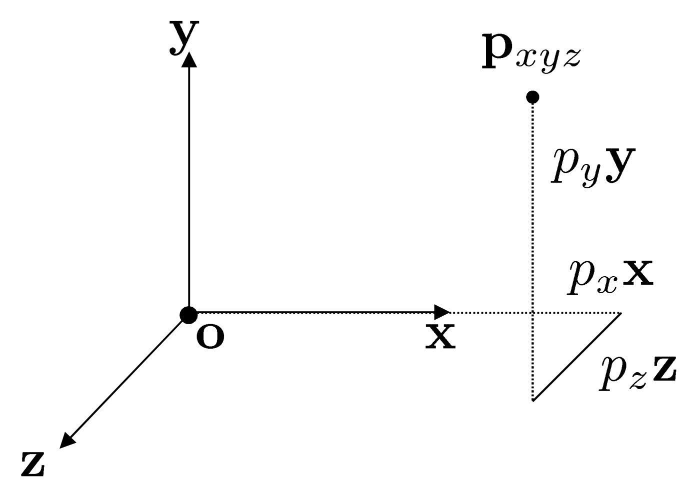
- Inverse: $$\begin{bmatrix} 1 & 0 & 0 & -t_x \\ 0 & 1 & 0 & -t_y \\ 0 & 0 & 1 & -t_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Today

- Coordinate Transformation
- Typical Coordinate Systems

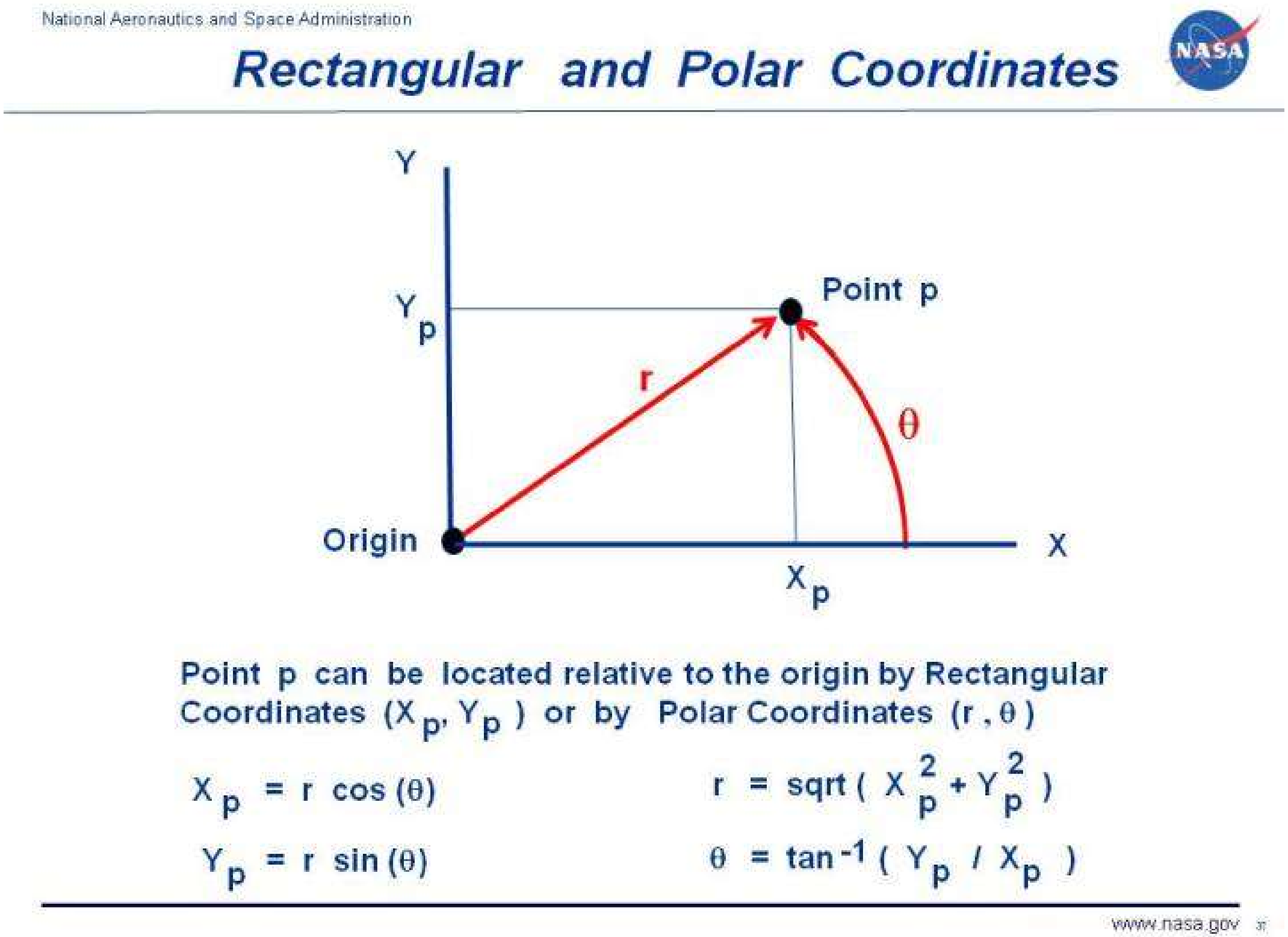
Coordinate System

- Given point **p** in homogeneous coordinates: $\begin{bmatrix} p_x \\ p_y \\ p_z \\ 1 \end{bmatrix}$
- Coordinates describe the point's 3D position in a coordinate system with basis vectors **x**, **y**, **z** and origin **o**:

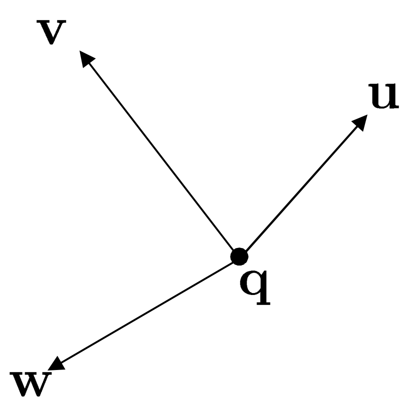


$$\mathbf{p}_{xyz} = p_x\mathbf{x} + p_y\mathbf{y} + p_z\mathbf{z} + \mathbf{o}$$

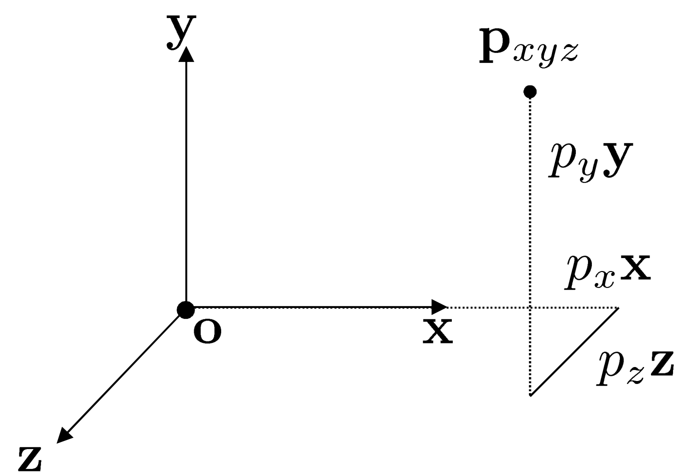
Rectangular and Polar Coordinates



Coordinate Transformation



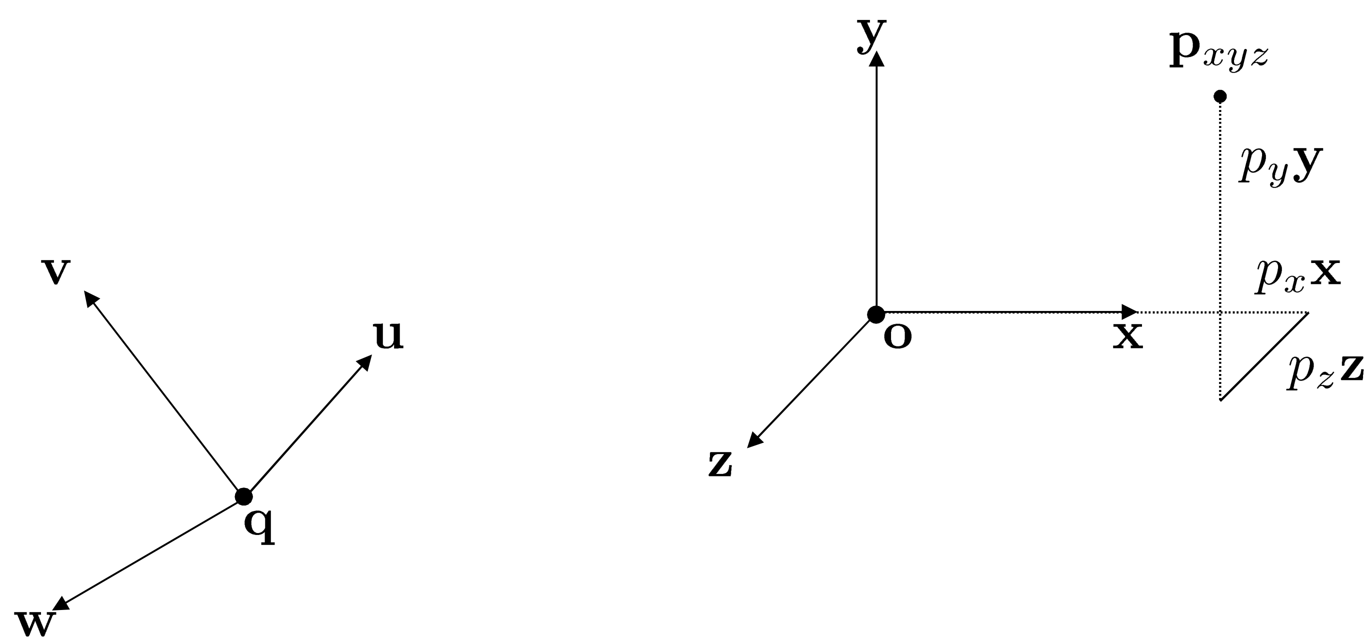
New **uvwq** coordinate system



Original **xyzo** coordinate system

Goal: Find coordinates of $\mathbf{p}_{xyz}$ in new **uvwq** coordinate system

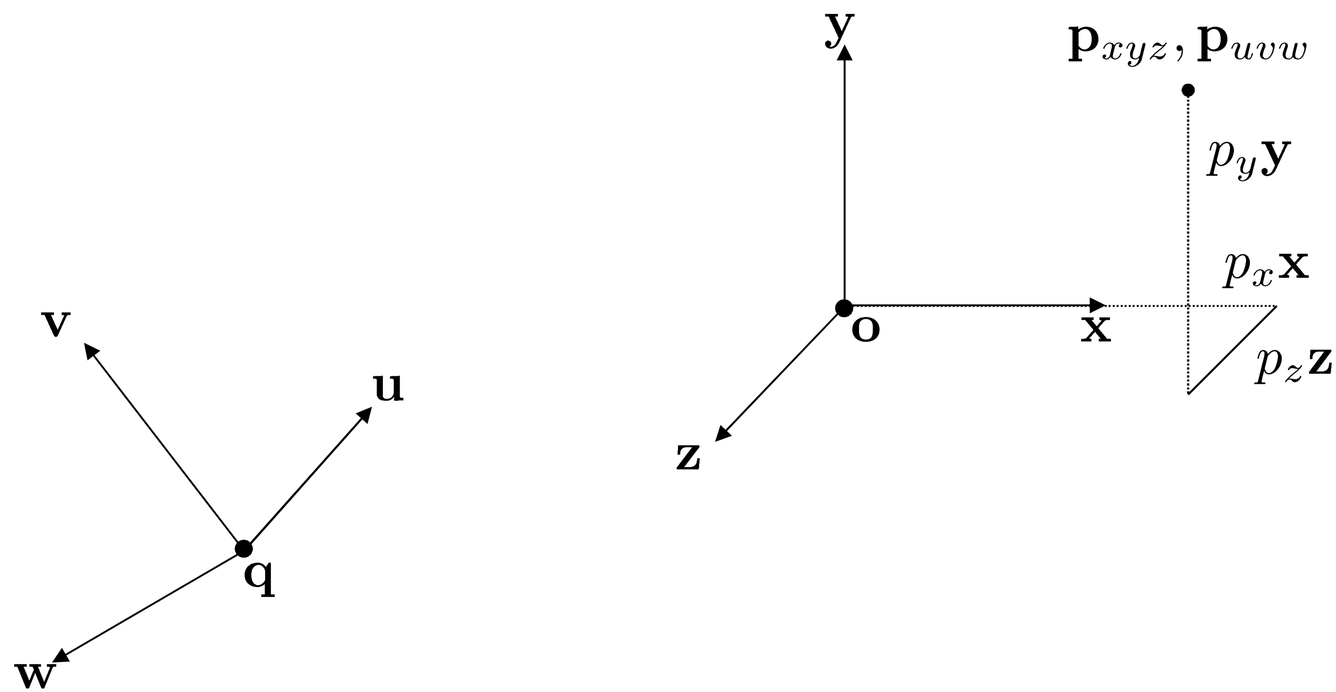
Coordinate Transformation



Express coordinates of **xyzo** reference frame with respect to **uvwq** reference frame:

$$\mathbf{x} = \begin{bmatrix} x_u \\ x_v \\ x_w \\ 0 \end{bmatrix} \qquad \mathbf{y} = \begin{bmatrix} y_u \\ y_v \\ y_w \\ 0 \end{bmatrix} \qquad \mathbf{z} = \begin{bmatrix} z_u \\ z_v \\ z_w \\ 0 \end{bmatrix} \qquad \mathbf{o} = \begin{bmatrix} o_u \\ o_v \\ o_w \\ 1 \end{bmatrix}$$

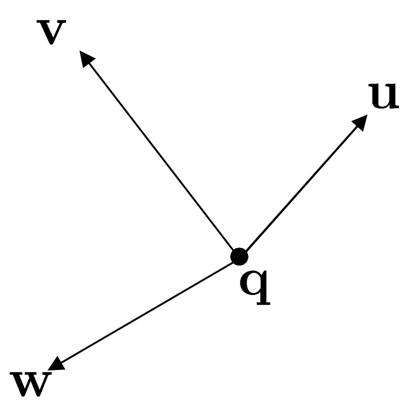
Coordinate Transformation

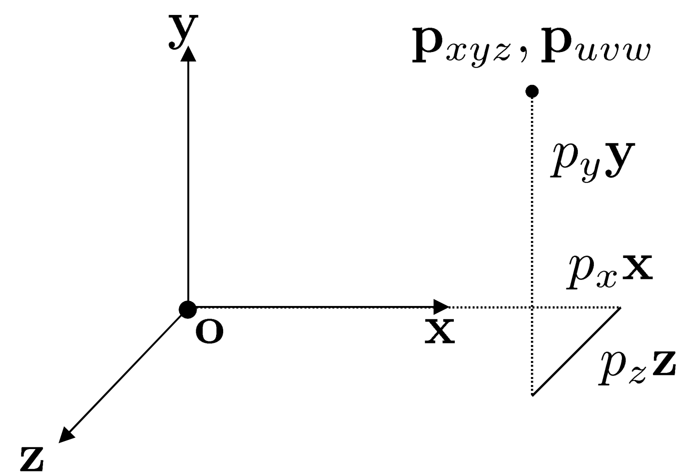


Point $\mathbf{p}$ expressed in new $\mathbf{uvwq}$ reference frame:

$$\mathbf{p}_{uvw} = p_x \begin{bmatrix} x_u \\ x_v \\ x_w \\ 0 \end{bmatrix} + p_y \begin{bmatrix} y_u \\ y_v \\ y_w \\ 0 \end{bmatrix} + p_z \begin{bmatrix} z_u \\ z_v \\ z_w \\ 0 \end{bmatrix} + \begin{bmatrix} o_u \\ o_v \\ o_w \\ 1 \end{bmatrix}$$

Coordinate Transformation





$$\mathbf{p}_{uvw} = \begin{bmatrix} x_u & y_u & z_u & o_u \\ x_v & y_v & z_v & o_v \\ x_w & y_w & z_w & o_w \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ p_z \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{x} & \mathbf{y} & \mathbf{z} & \mathbf{o} \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ p_z \\ 1 \end{bmatrix}$$

Coordinate Transformation

Inverse transformation

- Given point $\mathbf{P}_{uvw}$ w.r.t. reference frame **uvwq:**
  - Coordinates $\mathbf{P}_{xyz}$ w.r.t. reference frame **xyzo** are calculated as:

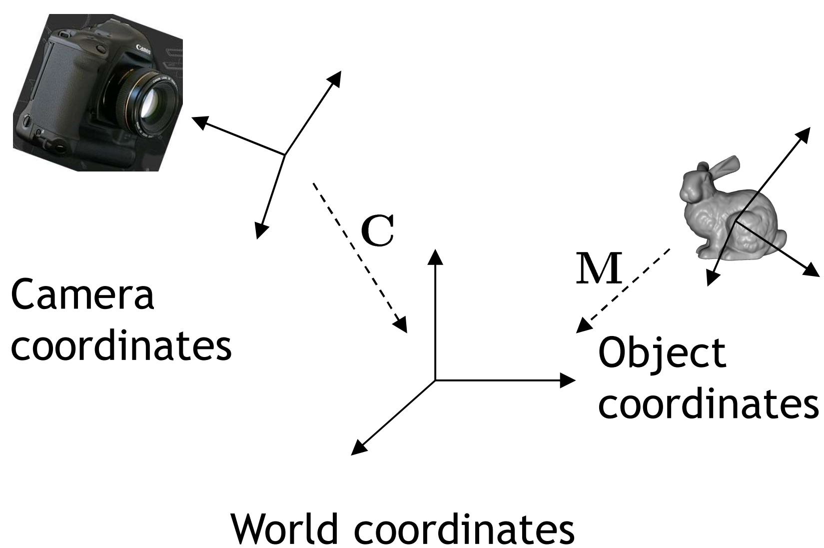
$$\mathbf{p}_{xyz} = \begin{bmatrix} x_u & y_u & z_u & o_u \\ x_v & y_v & z_v & o_v \\ x_w & y_w & z_w & o_w \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} p_u \\ p_v \\ p_w \\ 1 \end{bmatrix}$$

Lecture Overview

- Coordinate Transformation
- Typical Coordinate Systems

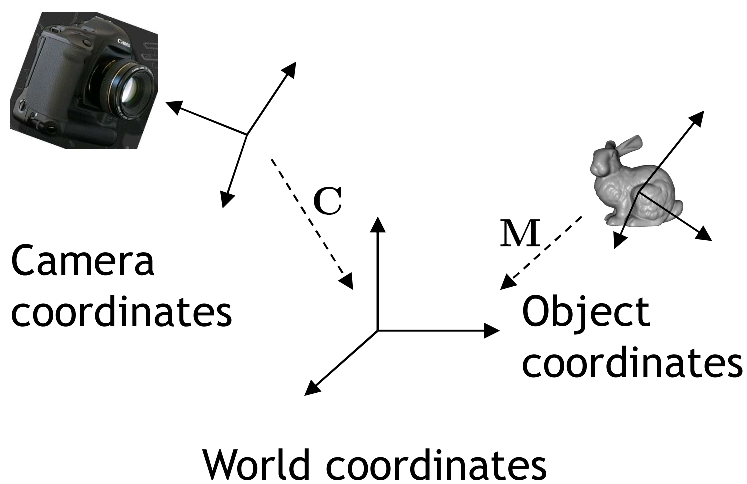
Typical Coordinate Systems

- In computer graphics, we typically use at least three coordinate systems:
  - World coordinate system
  - Camera coordinate system
  - Object coordinate system



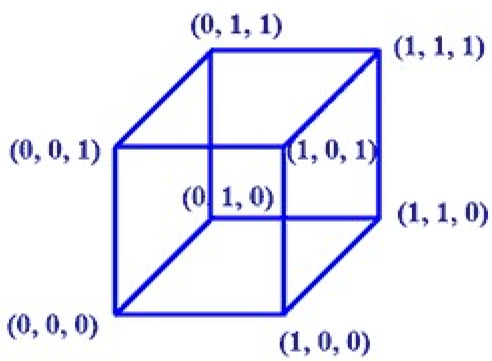
World Coordinates

- Common reference frame for all objects in the scene
- No standard for coordinate system orientation
  - If there is a ground plane, usually x/y is horizontal and z points up (height)
  - Otherwise, x/y is often screen plane, z points out of the screen

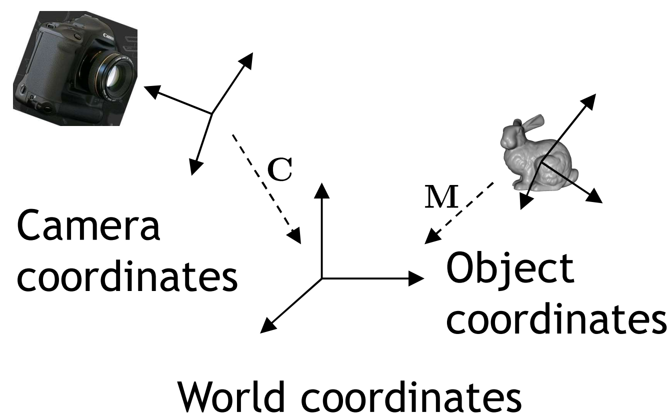


Object Coordinates

- Local coordinates in which points and other object geometry are given
- Often origin is in geometric center, on the base, or in a corner of the object
  - Depends on how object is generated or used.

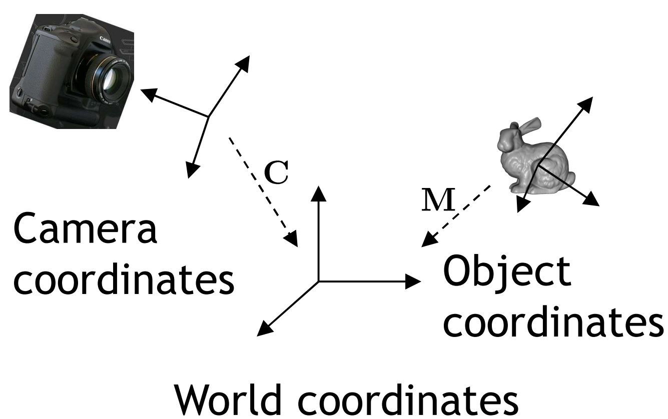


Source: http://motivate.maths.org



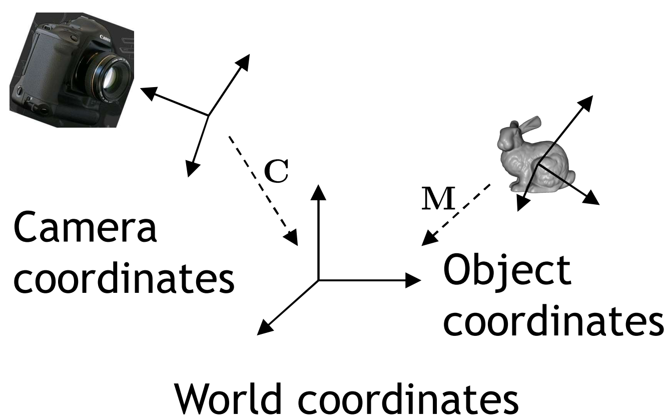
Object Transformation

- The transformation from object to world coordinates is different for each object.
- Defines placement of object in scene.
- Given by "model matrix" (model-to-world transformation) **M**.



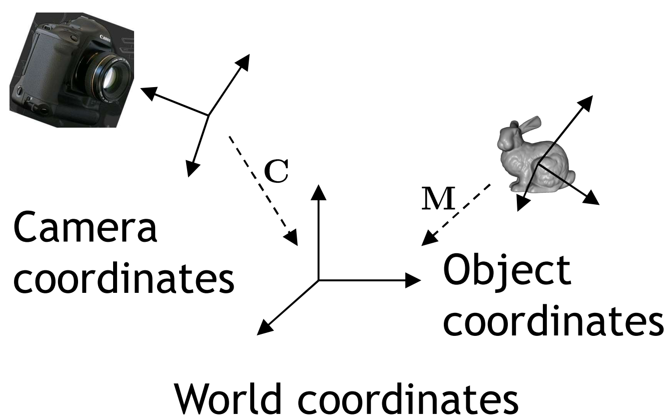
Camera Coordinate System

- Origin defines center of projection of camera
- x-y plane is parallel to image plane
- z-axis is perpendicular to image plane



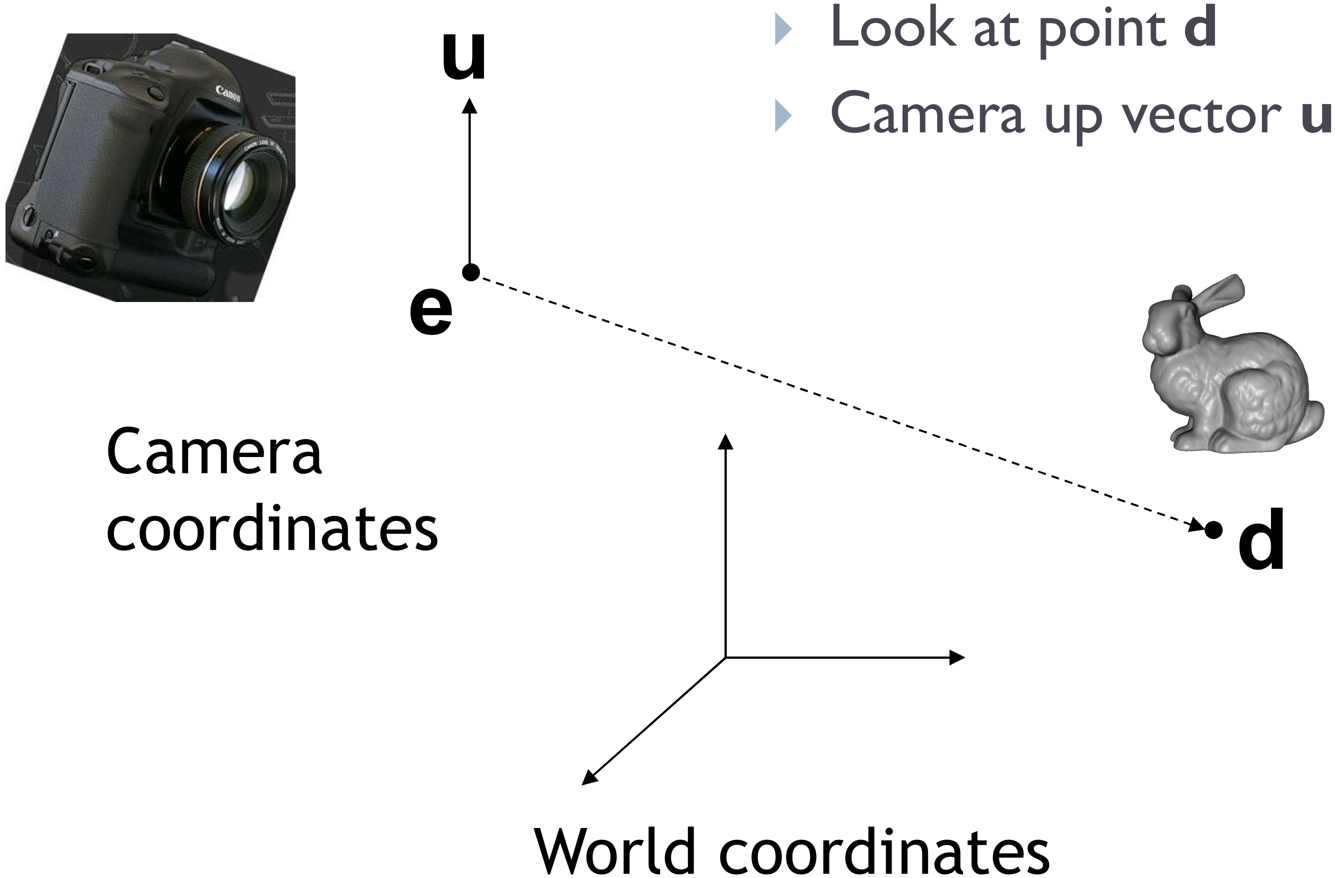
Camera Coordinate System

- The Camera Matrix defines the transformation from camera to world coordinates
  - Placement of camera in world



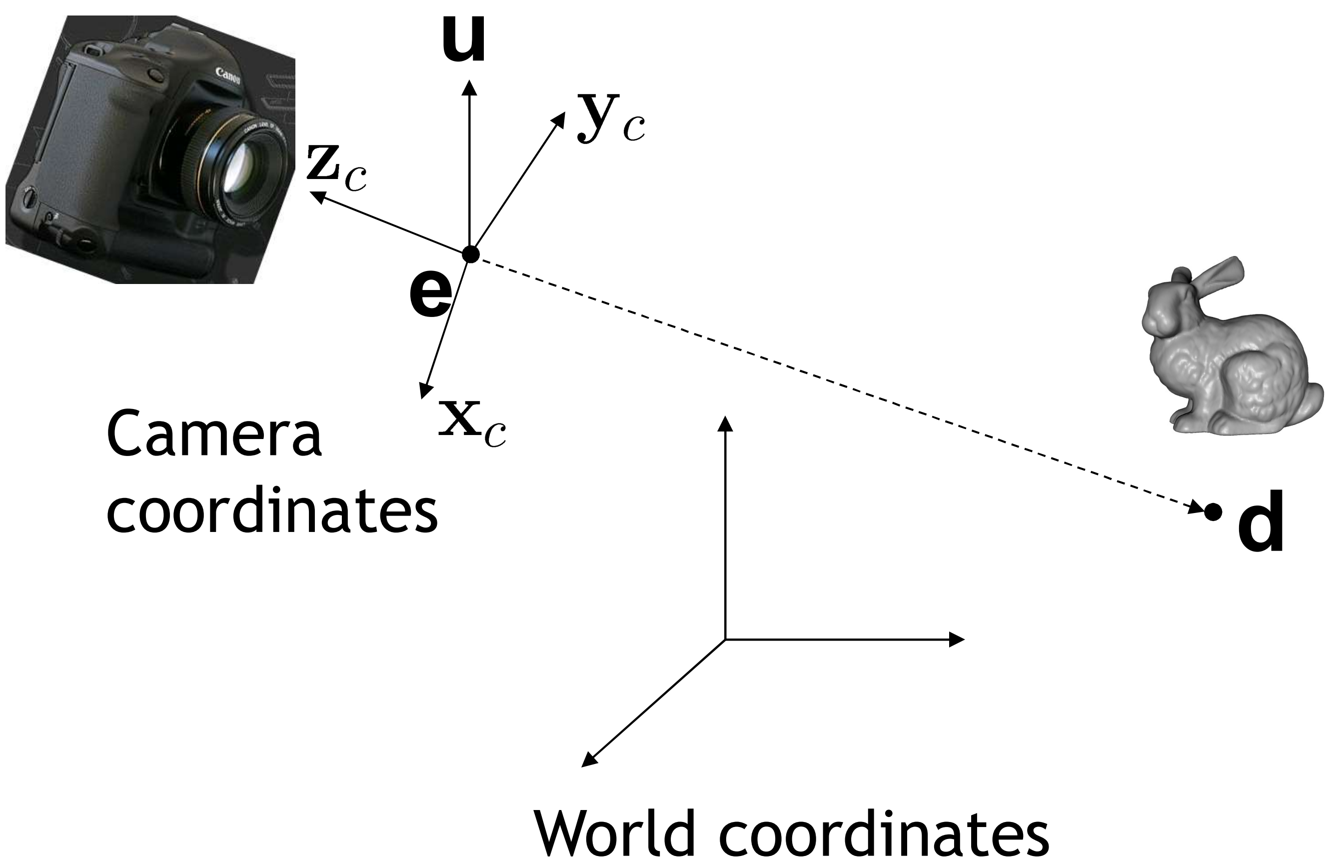
Camera Matrix

- Given:
  - Center point of projection **e**
  - Look at point **d**
  - Camera up vector **u**



Camera Matrix

- Construct $\mathbf{x}_c, \mathbf{y}_c, \mathbf{z}_c$

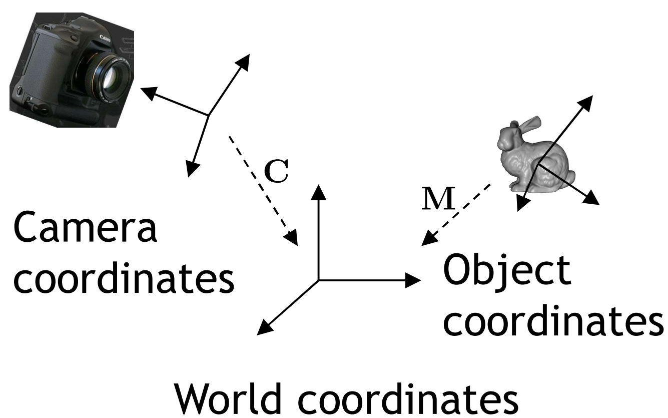


Camera Matrix

- Step 1: z-axis $$\boldsymbol{z}_C = \frac{\boldsymbol{e} - \boldsymbol{d}}{\|\boldsymbol{e} - \boldsymbol{d}\|}$$
- Step 2: x-axis $$\boldsymbol{x}_C = \frac{\boldsymbol{u} \times \boldsymbol{z}_C}{\|\boldsymbol{u} \times \boldsymbol{z}_C\|}$$
- Step 3: y-axis $$\boldsymbol{y}_C = \boldsymbol{z}_C \times \boldsymbol{x}_C = \frac{\boldsymbol{u}}{\|\boldsymbol{u}\|}$$
- Camera Matrix: $$\boldsymbol{C} = \begin{bmatrix} \boldsymbol{x}_C & \boldsymbol{y}_C & \boldsymbol{z}_C & \boldsymbol{e} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Transforming Object to Camera Coordinates

- Object to world coordinates: **M**
- Camera to world coordinates: **C**
- Point to transform: **p**
- Resulting transformation equation: $\mathbf{p' = C^{-1} M p}$



Tips for Notation

- Indicate coordinate systems with every point or matrix
  - Point: $\mathbf{p}_{\text{object}}$
  - Matrix: $\mathbf{M}_{\text{object}\rightarrow\text{world}}$
- Resulting transformation equation:

$$\mathbf{p}_{\text{camera}} = (\mathbf{C}_{\text{camera}\rightarrow\text{world}})^{-1}\ \mathbf{M}_{\text{object}\rightarrow\text{world}}\ \mathbf{p}_{\text{object}}$$

- In source code use similar names:
  - Point: `p_object` or `p_obj` or `p_o`
  - Matrix: `object2world` or `obj2wld` or `o2w`
- Resulting transformation equation:

```
wld2cam = inverse(cam2wld);
p_cam = p_obj * obj2wld * wld2cam;
```

Inverse of Camera Matrix

- How to calculate the inverse of camera matrix $\mathbf{C}^{-1}$?
- Generic matrix inversion is complex and compute-intensive!
- Solution: affine transformation matrices can be inverted more easily
- Observation:
  - Camera matrix consists of translation and rotation: $\mathbf{T} \times \mathbf{R}$
- Inverse of rotation: $\mathbf{R}^{-1} = \mathbf{R}^T$
- Inverse of translation: $\mathbf{T}(t)^{-1} = \mathbf{T}(-t)$
- Inverse of camera matrix: $\mathbf{C}^{-1} = \mathbf{R}^{-1} \times \mathbf{T}^{-1}$

Objects in Camera Coordinates

- We have things lined up the way we like them on screen
  - **x** points to the right
  - **y** points up
  - **-z** into the screen (i.e., **z** points out of the screen)
  - Objects to look at are in front of us, i.e., have negative z values
- But objects are still in 3D
- Next step: project scene to 2D plane