

CSE 167:
Introduction to Computer Graphics
Lecture #3: Projection

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Announcements

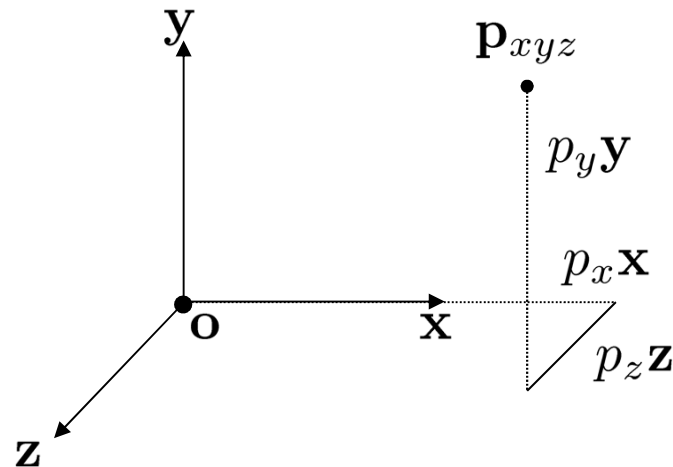
- ▶ **Project I due Friday at 2pm**
 - ▶ Grading window is 2-3:30pm
 - ▶ Upload source code to TritonEd by 2pm

Lecture Overview

- ▶ **Coordinate Transformation**
- ▶ Typical Coordinate Systems
- ▶ Projection

Coordinate System

- ▶ Given point **p** in homogeneous coordinates: $\begin{bmatrix} p_x \\ p_y \\ p_z \\ 1 \end{bmatrix}$
- ▶ Coordinates describe the point's 3D position in a coordinate system with basis vectors **x**, **y**, **z** and origin **o**:

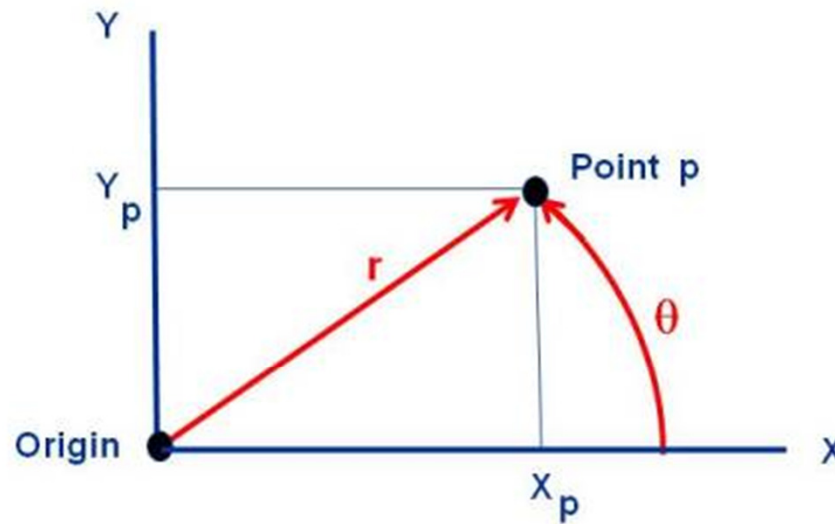


$$\mathbf{p}_{xyz} = p_x\mathbf{x} + p_y\mathbf{y} + p_z\mathbf{z} + \mathbf{o}$$

Rectangular and Polar Coordinates

National Aeronautics and Space Administration

Rectangular and Polar Coordinates



Point p can be located relative to the origin by Rectangular Coordinates (X_p, Y_p) or by Polar Coordinates (r, θ)

$$X_p = r \cos(\theta)$$

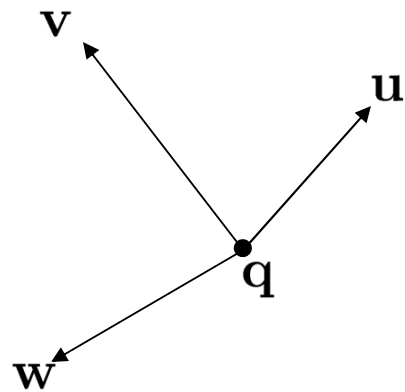
$$Y_p = r \sin(\theta)$$

$$r = \sqrt{X_p^2 + Y_p^2}$$

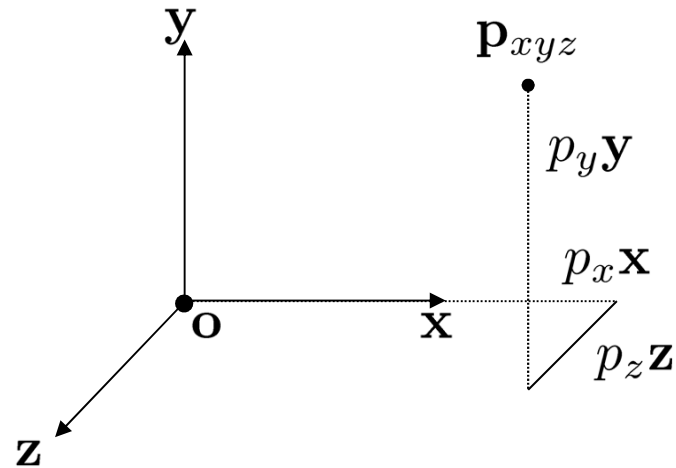
$$\theta = \tan^{-1}(Y_p / X_p)$$

www.nasa.gov

Coordinate Transformation



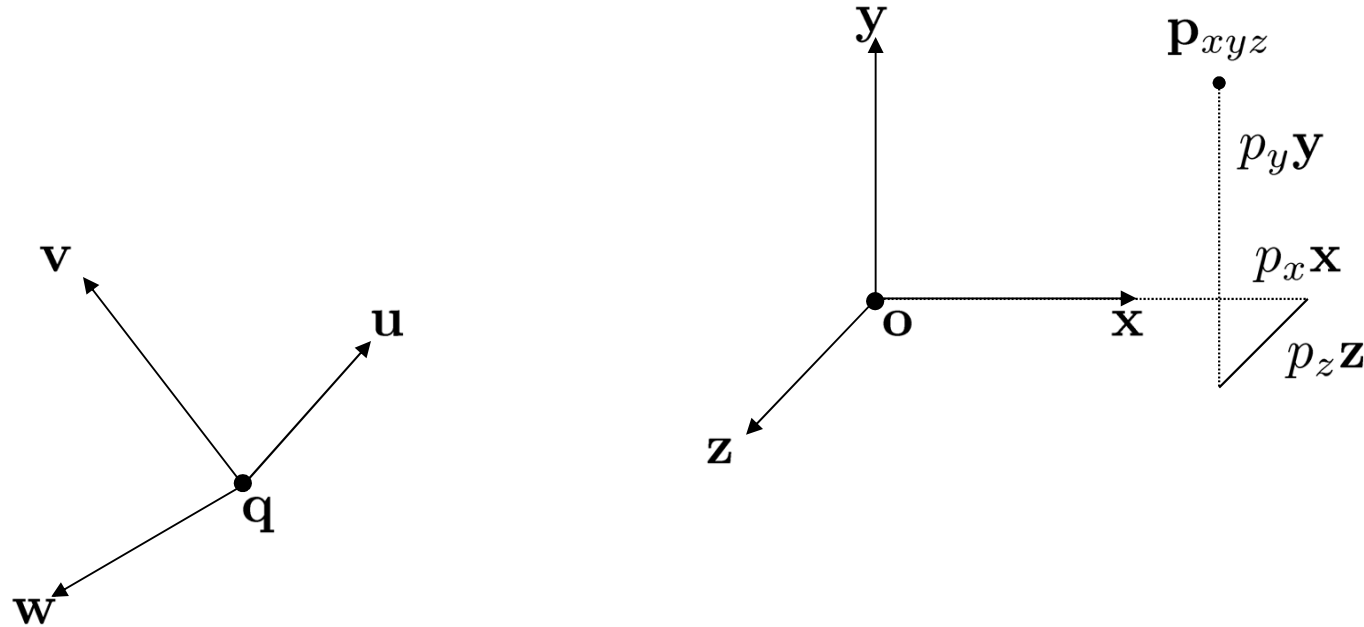
New **uvwq** coordinate system



Original **xyzo** coordinate system

Goal: Find coordinates of p_{xyz} in new **uvwq** coordinate system

Coordinate Transformation



Express coordinates of **xyzo** reference frame
with respect to **uvwq** reference frame:

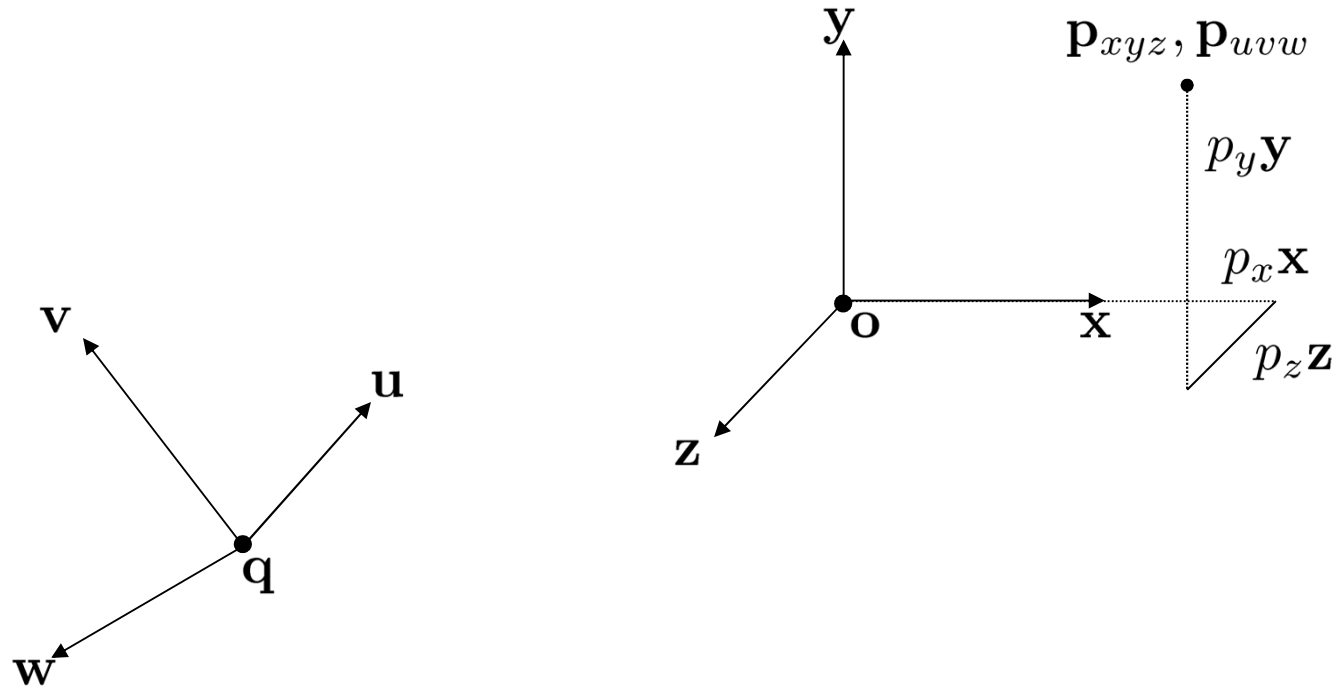
$$\mathbf{x} = \begin{bmatrix} x_u \\ x_v \\ x_w \\ 0 \end{bmatrix}$$

$$\mathbf{y} = \begin{bmatrix} y_u \\ y_v \\ y_w \\ 0 \end{bmatrix}$$

$$\mathbf{z} = \begin{bmatrix} z_u \\ z_v \\ z_w \\ 0 \end{bmatrix}$$

$$\mathbf{o} = \begin{bmatrix} o_u \\ o_v \\ o_w \\ 1 \end{bmatrix}$$

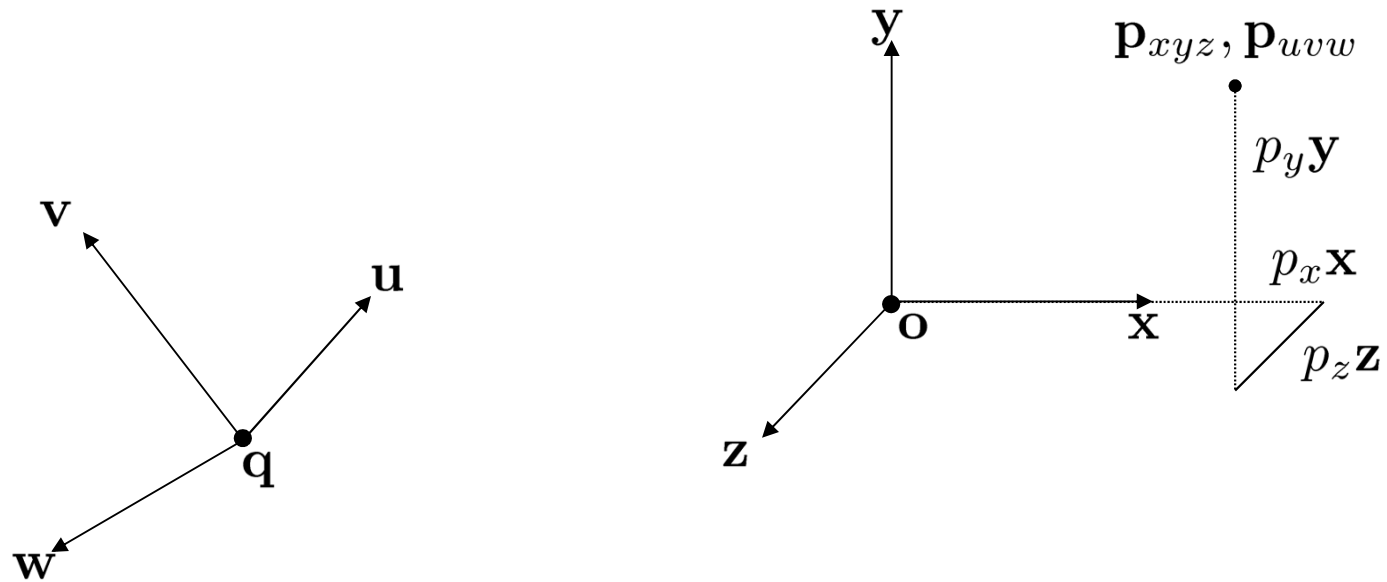
Coordinate Transformation



Point \mathbf{p} expressed in new \mathbf{uvw} reference frame:

$$\mathbf{p}_{uvw} = p_x \begin{bmatrix} x_u \\ x_v \\ x_w \\ 0 \end{bmatrix} + p_y \begin{bmatrix} y_u \\ y_v \\ y_w \\ 0 \end{bmatrix} + p_z \begin{bmatrix} z_u \\ z_v \\ z_w \\ 0 \end{bmatrix} + \begin{bmatrix} o_u \\ o_v \\ o_w \\ 1 \end{bmatrix}$$

Coordinate Transformation



$$\mathbf{p}_{uvw} = \begin{bmatrix} x_u & y_u & z_u & o_u \\ x_v & y_v & z_v & o_v \\ x_w & y_w & z_w & o_w \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ p_z \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{x} & \mathbf{y} & \mathbf{z} & \mathbf{o} \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ p_z \\ 1 \end{bmatrix}$$

Coordinate Transformation

Inverse transformation

- ▶ Given point \mathbf{P}_{uvw} w.r.t. reference frame **uvwq**:
 - ▶ Coordinates \mathbf{P}_{xyz} w.r.t. reference frame **xyzo** are calculated as:

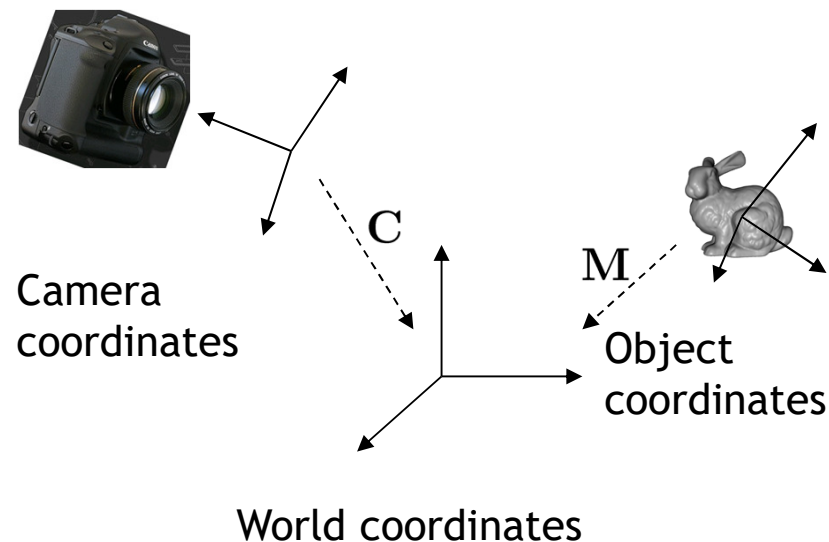
$$\mathbf{P}_{xyz} = \begin{bmatrix} x_u & y_u & z_u & o_u \\ x_v & y_v & z_v & o_v \\ x_w & y_w & z_w & o_w \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} p_u \\ p_v \\ p_w \\ 1 \end{bmatrix}$$

Lecture Overview

- ▶ Concatenating Transformations
- ▶ Coordinate Transformation
- ▶ **Typical Coordinate Systems**
- ▶ Projection

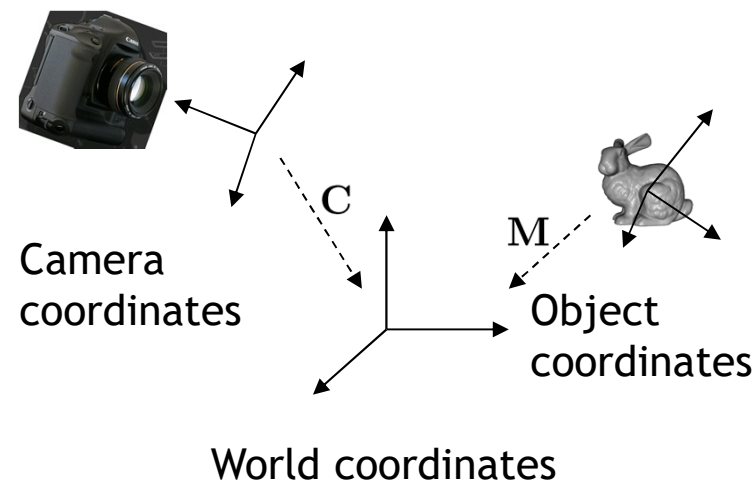
Typical Coordinate Systems

- ▶ In computer graphics, we typically use at least three coordinate systems:
 - ▶ World coordinate system
 - ▶ Camera coordinate system
 - ▶ Object coordinate system



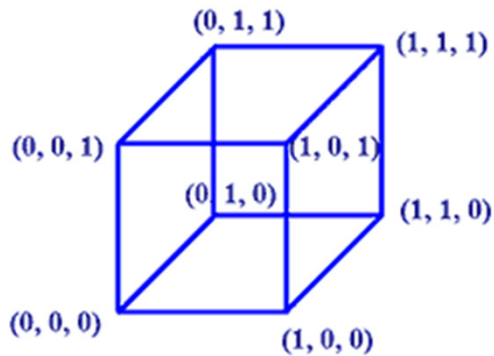
World Coordinates

- ▶ Common reference frame for all objects in the scene
- ▶ No standard for coordinate system orientation
 - ▶ If there is a ground plane, usually x/y is horizontal and z points up (height)
 - ▶ Otherwise, x/y is often screen plane, z points out of the screen

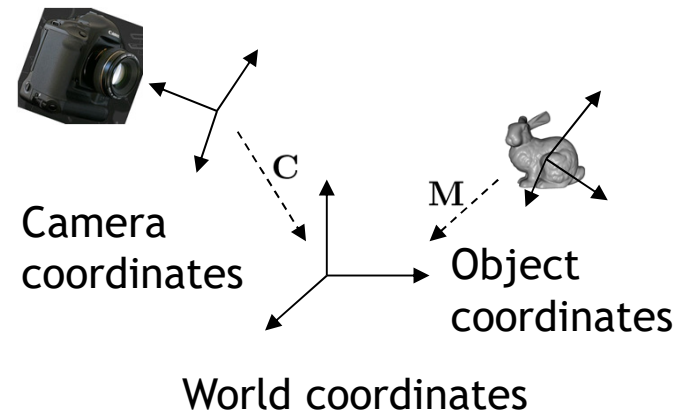


Object Coordinates

- ▶ Local coordinates in which points and other object geometry are given
- ▶ Often origin is in geometric center, on the base, or in a corner of the object
 - ▶ Depends on how object is generated or used.

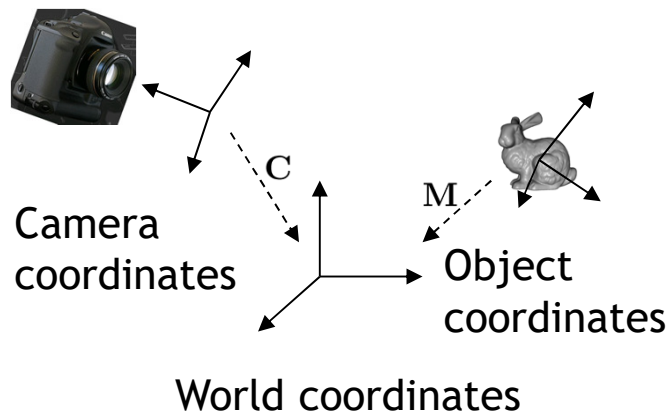


Source: <http://motivate.maths.org>



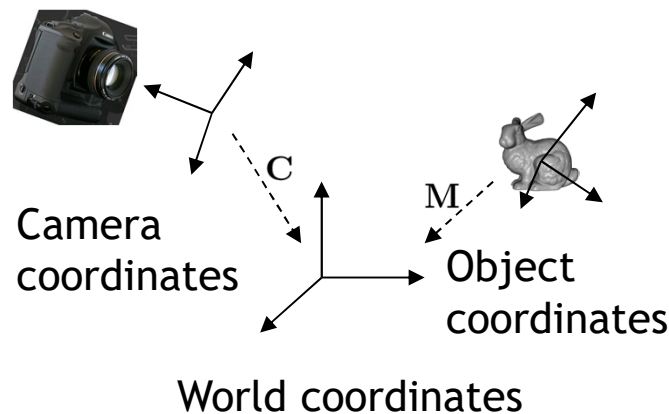
Object Transformation

- ▶ The transformation from object to world coordinates is different for each object.
- ▶ Defines placement of object in scene.
- ▶ Given by “model matrix” (model-to-world transformation) **M**.



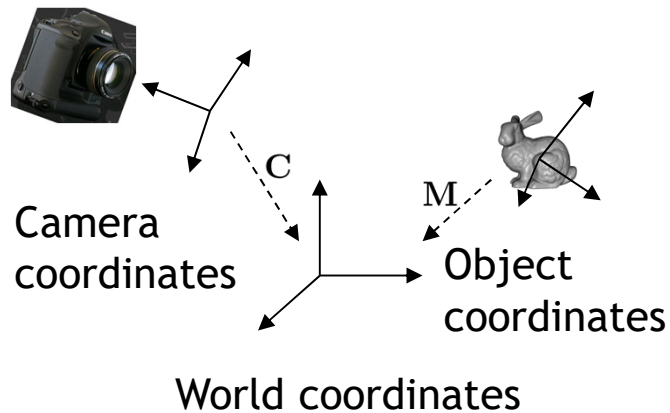
Camera Coordinate System

- ▶ Origin defines center of projection of camera
- ▶ x-y plane is parallel to image plane
- ▶ z-axis is perpendicular to image plane



Camera Coordinate System

- ▶ The Camera Matrix defines the transformation from camera to world coordinates
 - ▶ Placement of camera in world



Camera Matrix

- Construct from center of projection \mathbf{e} , look at \mathbf{d} , up-vector \mathbf{up} :



Camera
coordinates

\mathbf{up}
 \mathbf{e}

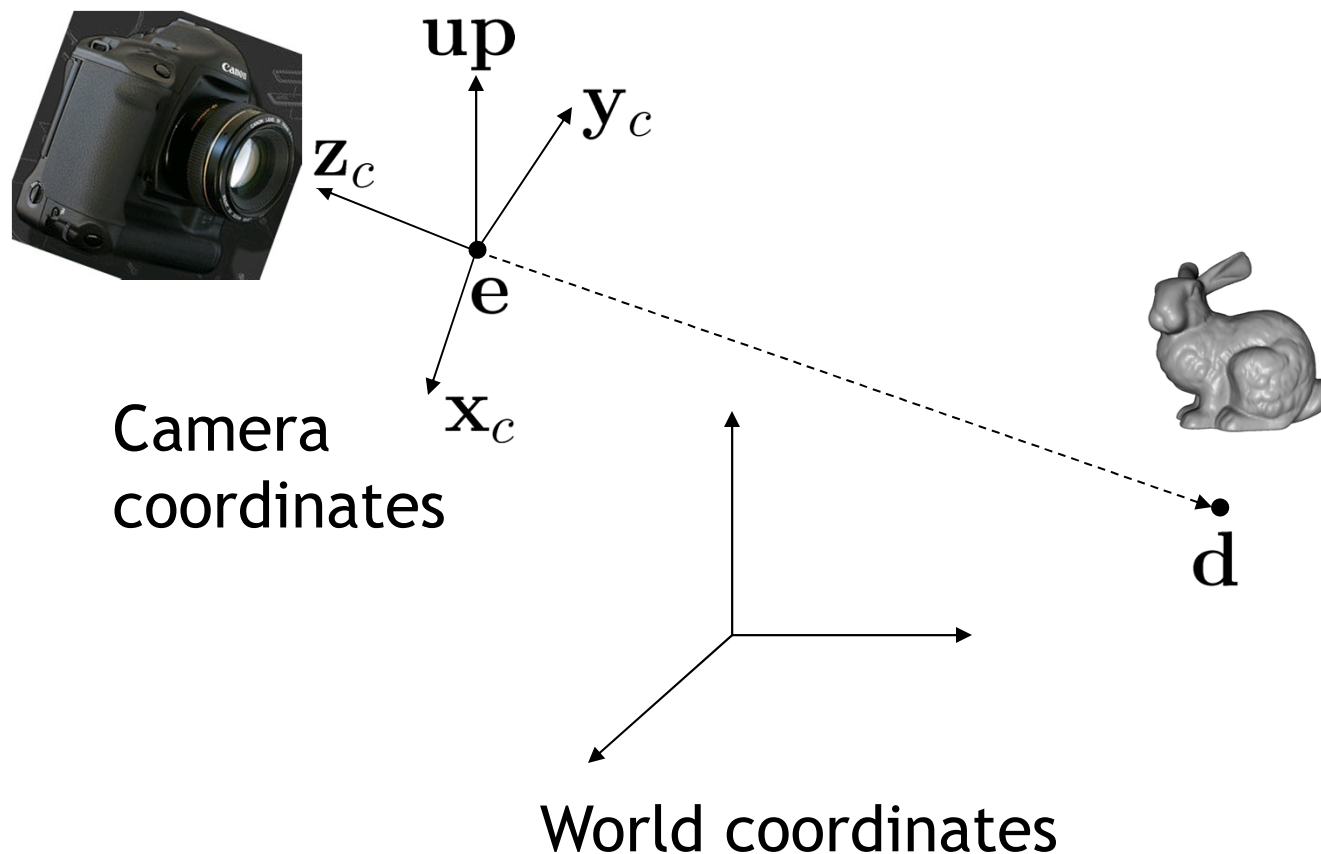


\mathbf{d}

World coordinates

Camera Matrix

- Construct from center of projection \mathbf{e} , look at \mathbf{d} , up-vector \mathbf{up} (up in camera coordinate system):



Camera Matrix

► **z-axis**

$$\mathbf{z}_C = \frac{\mathbf{e} - \mathbf{d}}{\|\mathbf{e} - \mathbf{d}\|}$$

► **x-axis**

$$\mathbf{x}_C = \frac{\mathbf{up} \times \mathbf{z}_C}{\|\mathbf{up} \times \mathbf{z}_C\|}$$

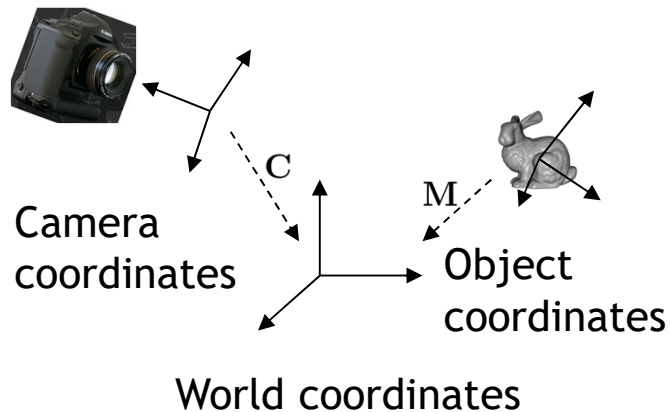
► **y-axis**

$$\mathbf{y}_C = \mathbf{z}_C \times \mathbf{x}_C = \frac{\mathbf{up}}{\|\mathbf{up}\|}$$

$$\mathbf{C} = \begin{bmatrix} \mathbf{x}_C & \mathbf{y}_C & \mathbf{z}_C & \mathbf{e} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Transforming Object to Camera Coordinates

- ▶ Object to world coordinates: **M**
- ▶ Camera to world coordinates: **C**
- ▶ Point to transform: **p**
- ▶ Resulting transformation equation: **$p' = C^{-1} M p$**



Tips for Notation

- ▶ Indicate coordinate systems with every point or matrix

- ▶ Point: $\mathbf{p}_{\text{object}}$

- ▶ Matrix: $\mathbf{M}_{\text{object} \rightarrow \text{world}}$

- ▶ Resulting transformation equation:

$$\mathbf{p}_{\text{camera}} = (\mathbf{C}_{\text{camera} \rightarrow \text{world}})^{-1} \mathbf{M}_{\text{object} \rightarrow \text{world}} \mathbf{p}_{\text{object}}$$

- ▶ Helpful hint: in source code use consistent names

- ▶ Point: `p_object` or `p_obj` or `p_o`

- ▶ Matrix: `object2world` or `obj2wld` or `o2w`

- ▶ Resulting transformation equation:

```
wld2cam = inverse(cam2wld);
```

```
p_cam = p_obj * obj2wld * wld2cam;
```

Inverse of Camera Matrix

- ▶ How to calculate the inverse of the camera matrix \mathbf{C}^{-1} ?
- ▶ Generic matrix inversion is complex and compute-intensive
- ▶ Solution: affine transformation matrices can be inverted more easily
- ▶ Observation:
 - ▶ Camera matrix consists of translation and rotation: $\mathbf{T} \times \mathbf{R}$
- ▶ Inverse of rotation: $\mathbf{R}^{-1} = \mathbf{R}^T$
- ▶ Inverse of translation: $\mathbf{T}(t)^{-1} = \mathbf{T}(-t)$
- ▶ Inverse of camera matrix: $\mathbf{C}^{-1} = \mathbf{R}^{-1} \times \mathbf{T}^{-1}$

Objects in Camera Coordinates

- ▶ We have things lined up the way we like them on screen
 - ▶ x to the right
 - ▶ y up
 - ▶ $-z$ into the screen
 - ▶ Objects to look at are in front of us, i.e. have negative z values
- ▶ But objects are still in 3D
- ▶ Next step: project scene to 2D plane

Lecture Overview

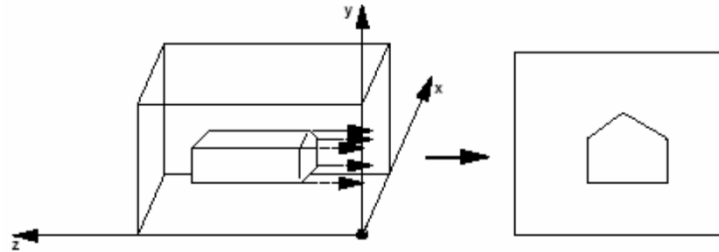
- ▶ Concatenating Transformations
- ▶ Coordinate Transformation
- ▶ Typical Coordinate Systems
- ▶ **Projection**

Projection

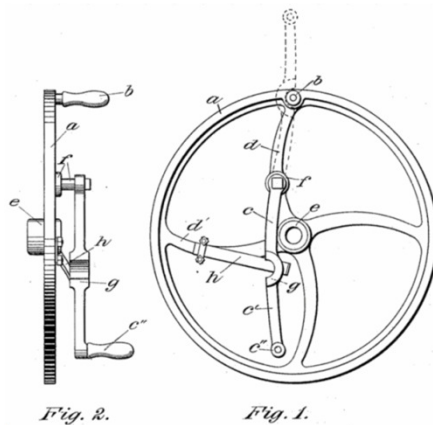
- ▶ **Goal:**
Given 3D points (vertices) in camera coordinates, determine corresponding image coordinates
- ▶ Transforming 3D points into 2D is called Projection
- ▶ OpenGL supports two types of projection:
 - ▶ Orthographic Projection (=Parallel Projection)
 - ▶ Perspective Projection

Orthographic Projection

- ▶ Can be done by ignoring **z**-coordinate
 - ▶ Use camera space **xy** coordinates as image coordinates
- ▶ Project points to **x-y** plane along parallel lines

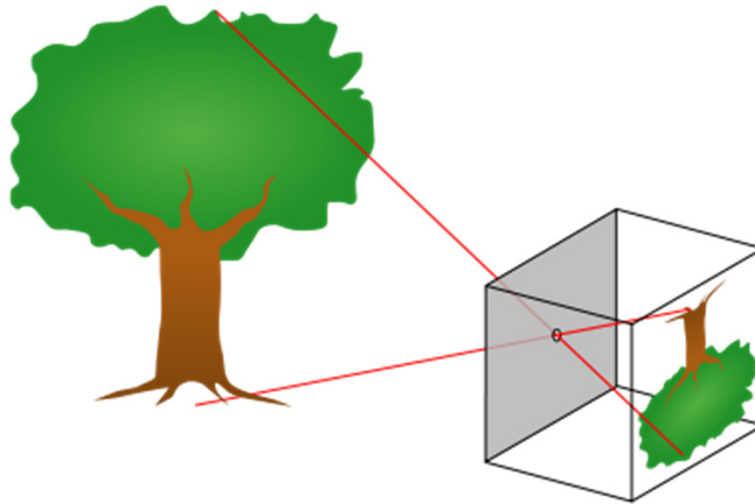


- ▶ Often used in graphical illustrations, architecture, 3D modeling



Perspective Projection

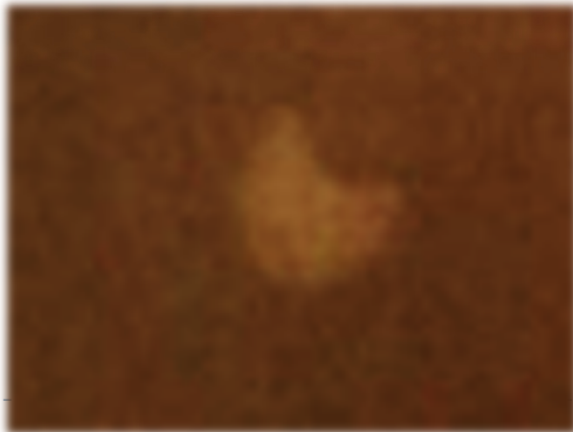
- ▶ Most common for computer graphics
- ▶ Simplified model of human eye, or camera lens (*pinhole camera*)



- ▶ Things farther away appear to be smaller
- ▶ Discovery attributed to Filippo Brunelleschi (Italian architect) in the early 1400's

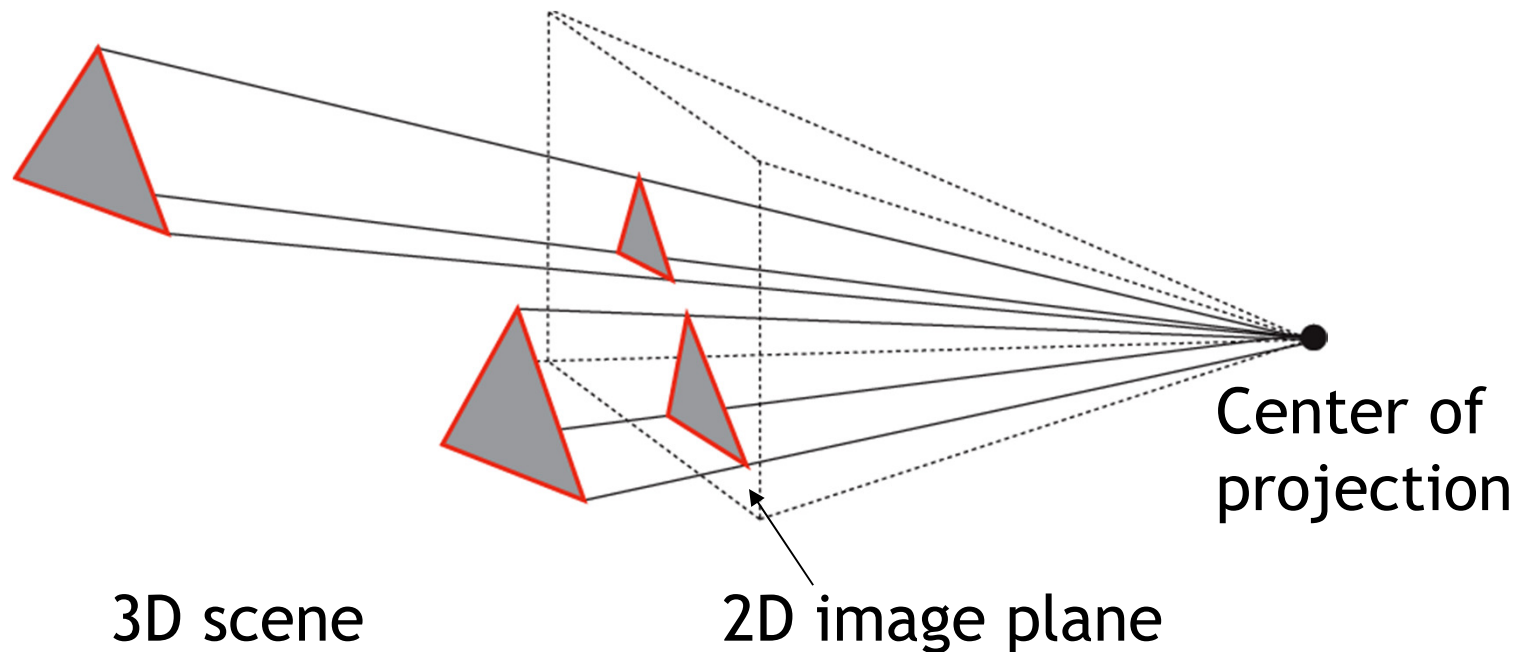
Pinhole Camera

- ▶ San Diego, May 20th, 2012



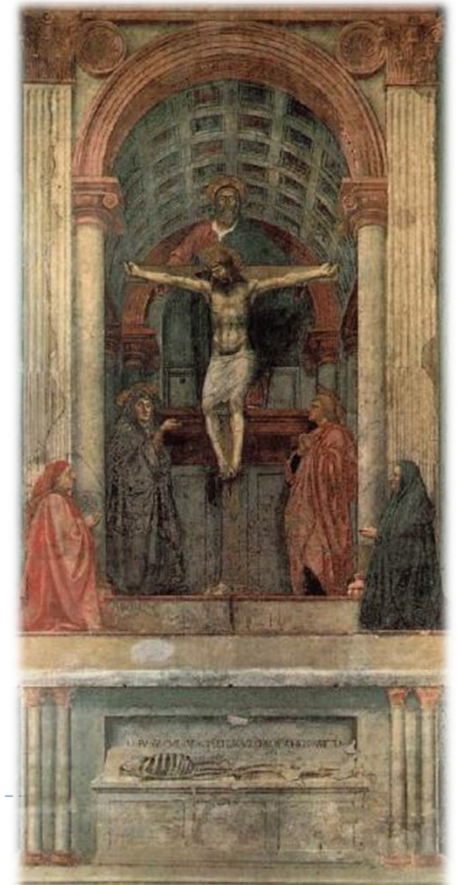
Perspective Projection

- Project along rays that converge in center of projection



Perspective Projection

Parallel lines are no longer parallel, converge in one point



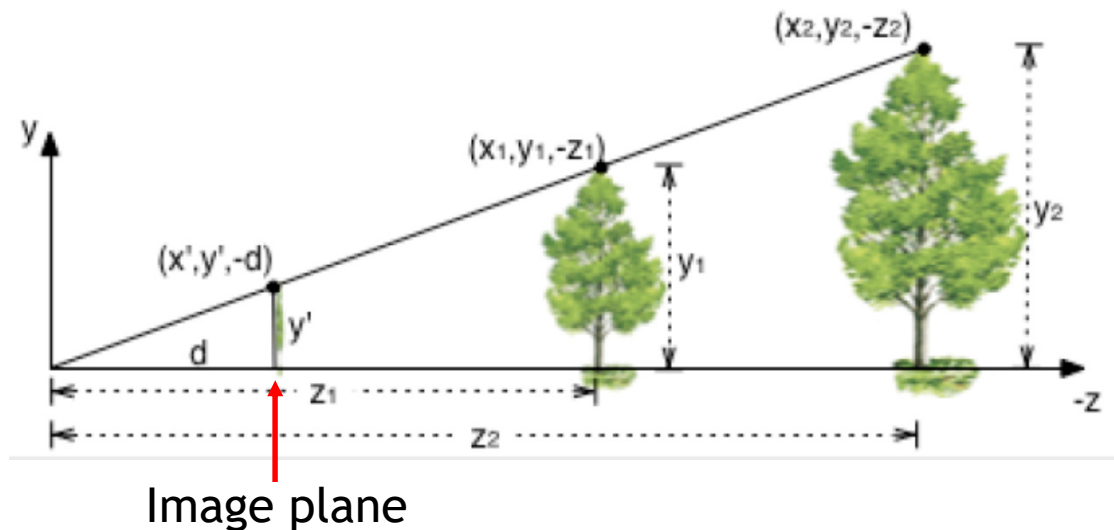
Earliest example:
La Trinità (1427) by Masaccio

Perspective Projection

From law of ratios in similar triangles follows:

$$\frac{y'}{d} = \frac{y_1}{z_1} \rightarrow y' = \frac{y_1 d}{z_1}$$

Similarly: $x' = \frac{x_1 d}{z_1}$



By definition: $z' = d$

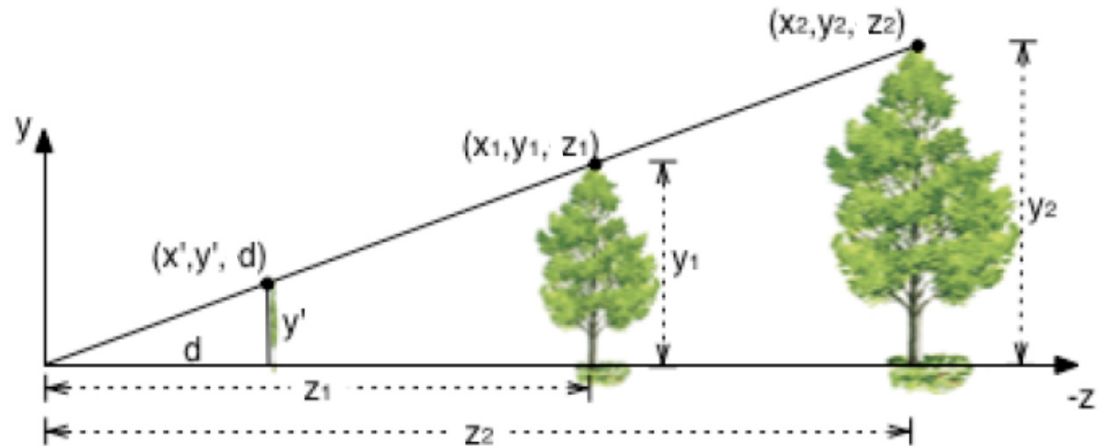
- ▶ We can express this using homogeneous coordinates and 4x4 matrices as follows

Perspective Projection

$$x' = \frac{x_1 d}{z_1}$$

$$y' = \frac{y_1 d}{z_1}$$

$$z' = d$$



$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1/d & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \\ z/d \end{bmatrix} \rightarrow \begin{bmatrix} xd/z \\ yd/z \\ d \\ 1 \end{bmatrix}$$

Projection matrix

Homogeneous division

Perspective Projection

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1/d & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \\ z/d \end{bmatrix} = \begin{bmatrix} xd/z \\ yd/z \\ d \\ 1 \end{bmatrix}$$

Projection matrix P

- ▶ Using projection matrix, homogeneous division seems more complicated than just multiplying all coordinates by d/z , so why do it?
- ▶ It will allow us to:
 - ▶ Handle different types of projections in a unified way
 - ▶ Define arbitrary view volumes