CSE 167:
Introduction to Computer Graphics
Lecture #10: Polynomial Curves

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Announcements

▸ Project 3 due this Friday

▸ Midterm #2 next Thursday

▸ No office hour today

  ▸ Moving it to Thursday 3:30-4:30pm

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Lecture Overview

- Polynomial Curves
  - Introduction
  - Polynomial functions

- Bézier Curves
  - Introduction
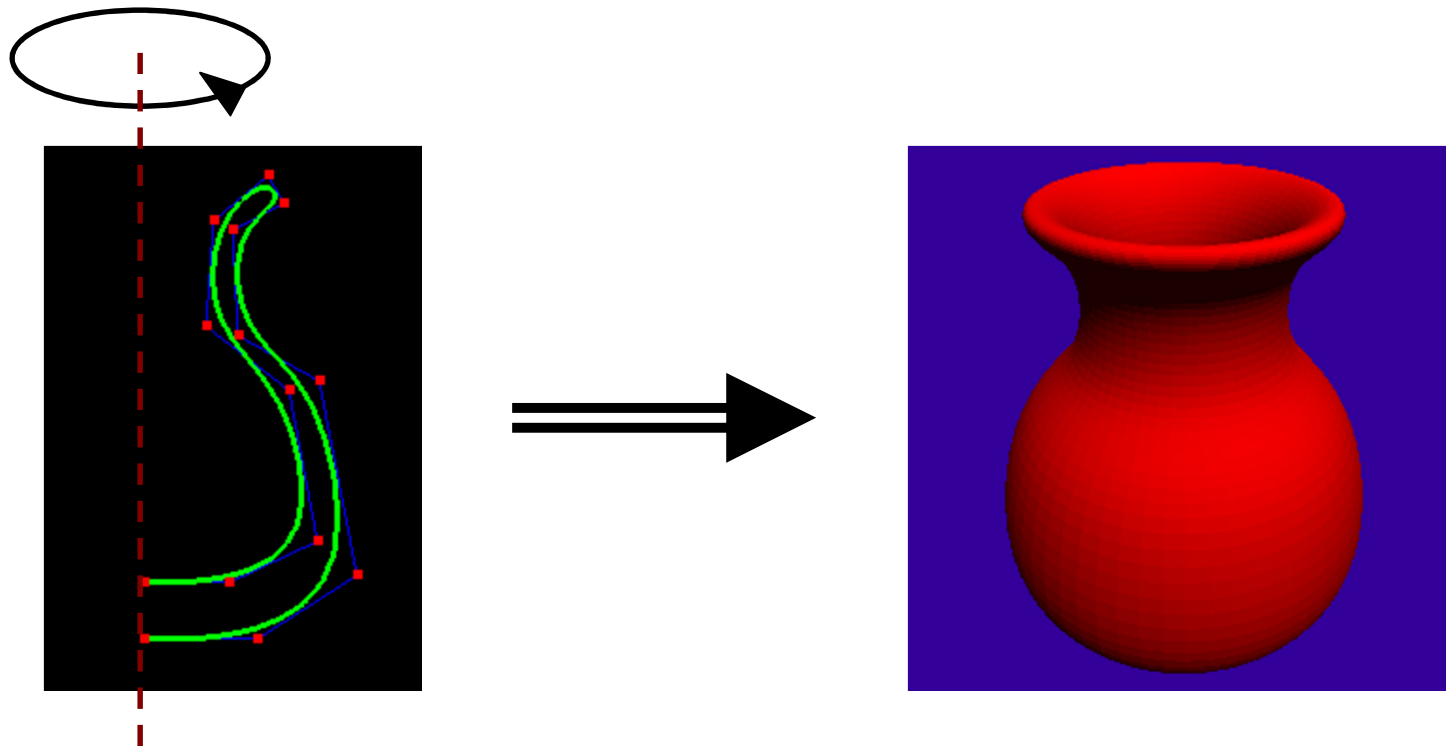 - Drawing Bézier curves
  - Piecewise Bézier curves

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Modeling

- Creating 3D objects

- How to construct complex surfaces?

- Goal
  - Specify objects with control points
  - Objects should be visually pleasing (smooth)

- Start with curves, then generalize to surfaces
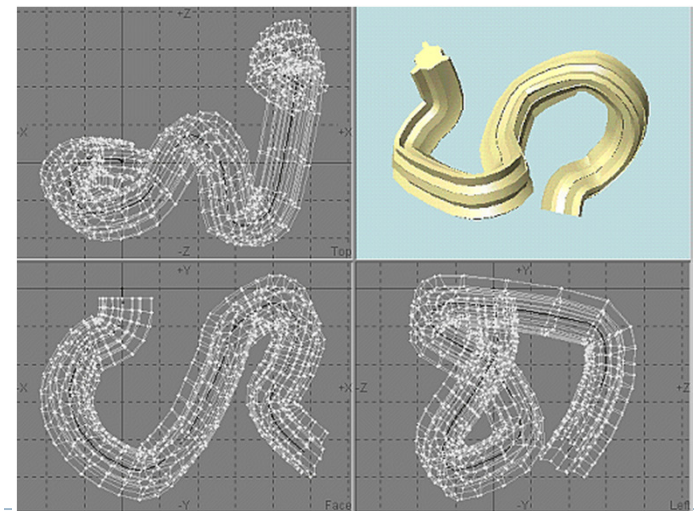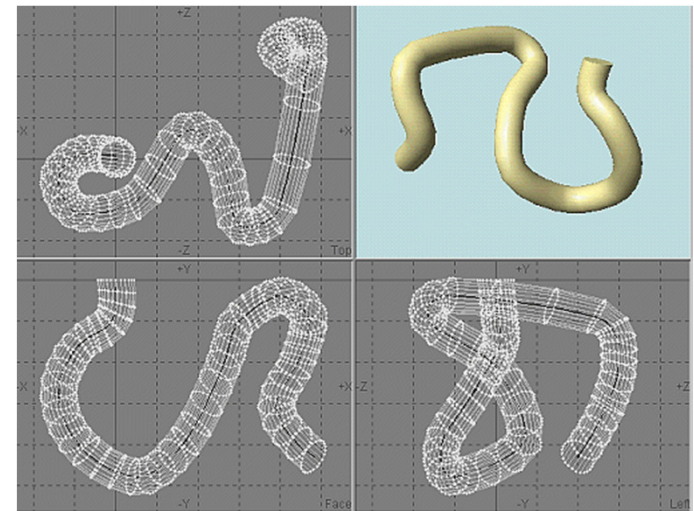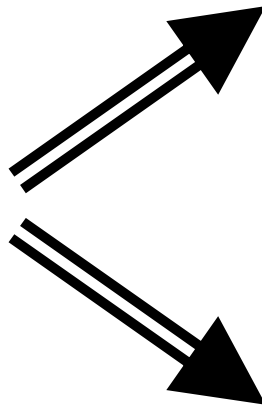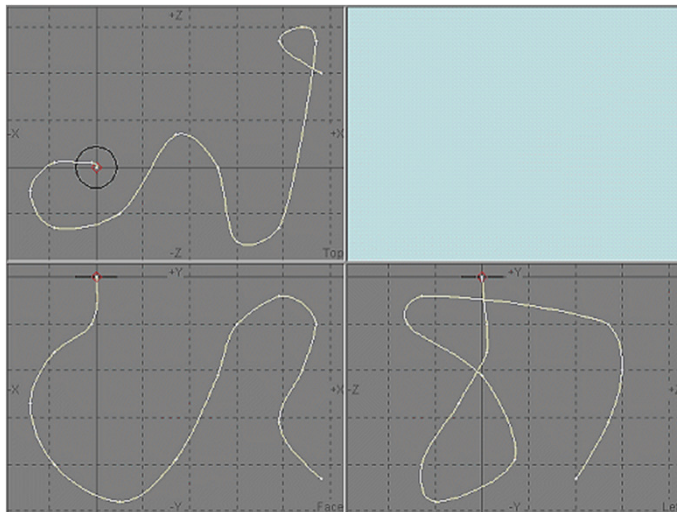
- Next: What can curves be used for?
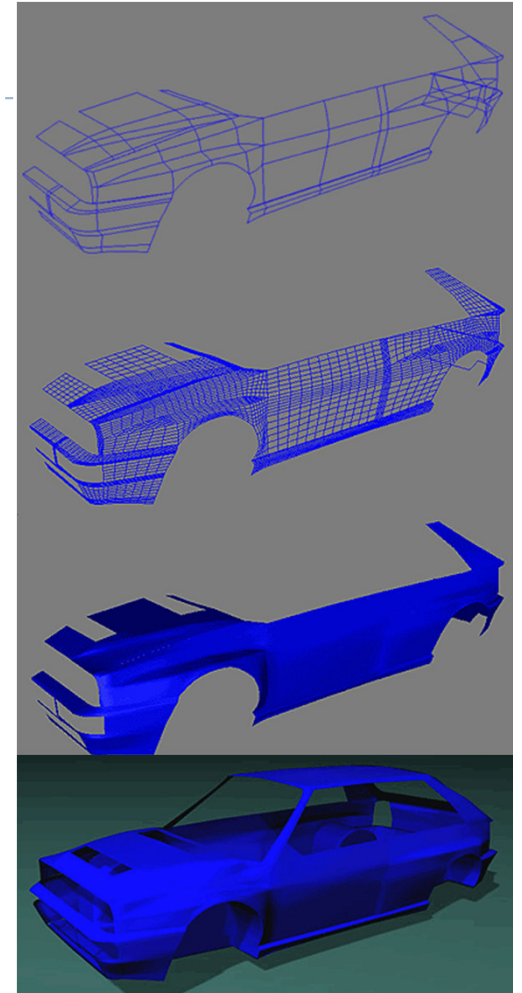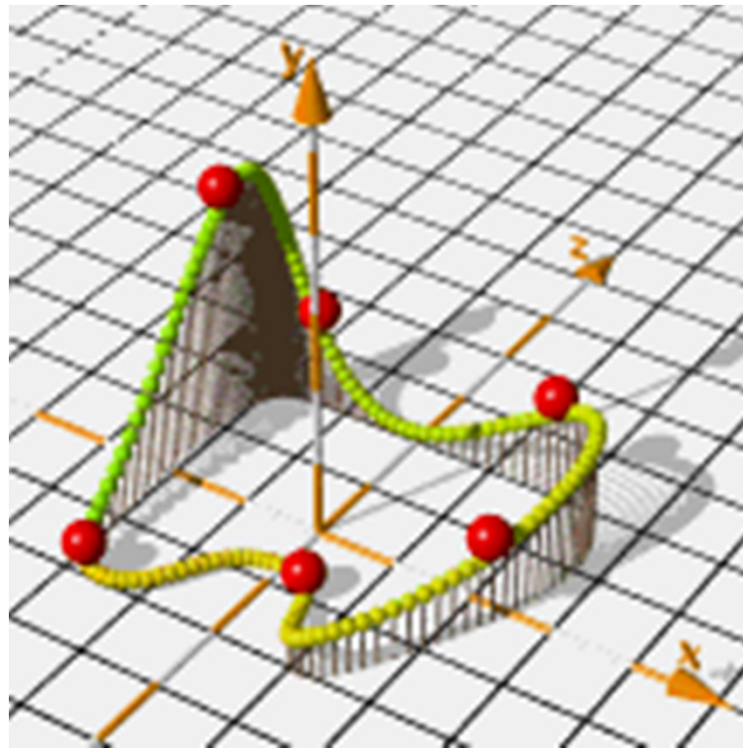
Curves

▸ Surface of revolution

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Curves

▸ Extruded/swept surfaces

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Curves

- Animation
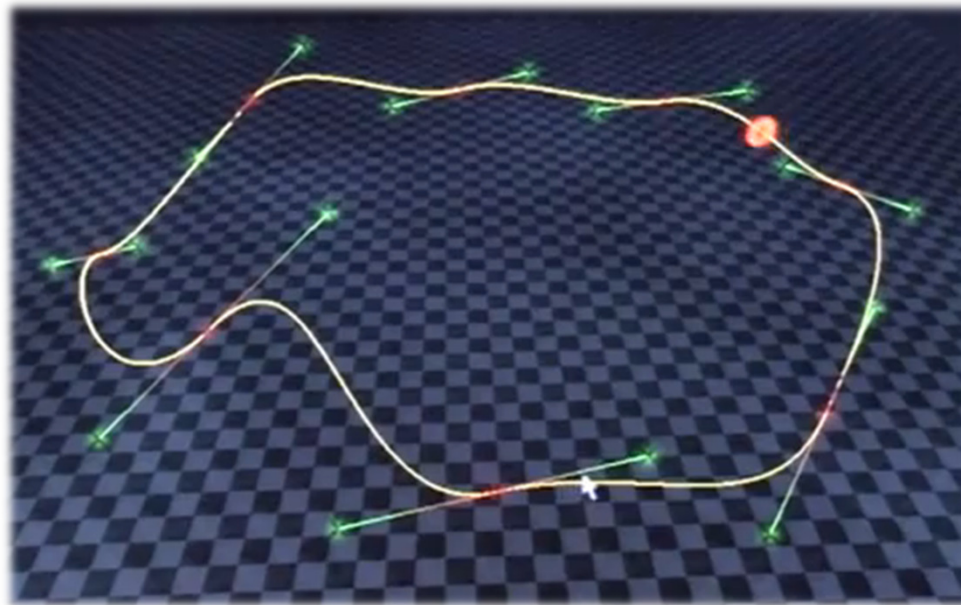 - Provide a "track" for objects
  - Use as camera path

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Video

- ## Bezier Curves
  - http://www.youtube.com/watch?v=hIDYJNEiYvU

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Curves

- Can be generalized to surface patches

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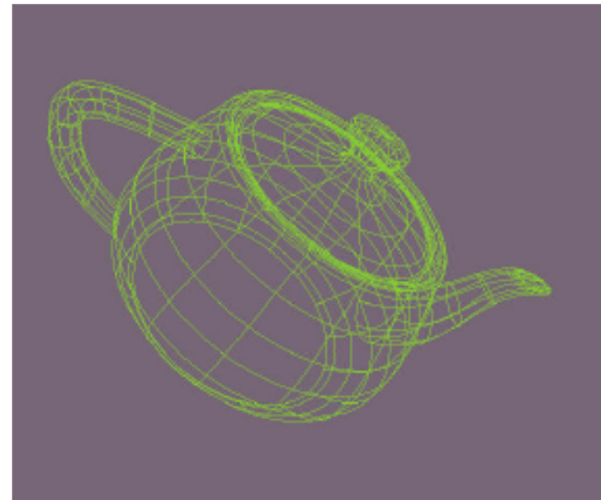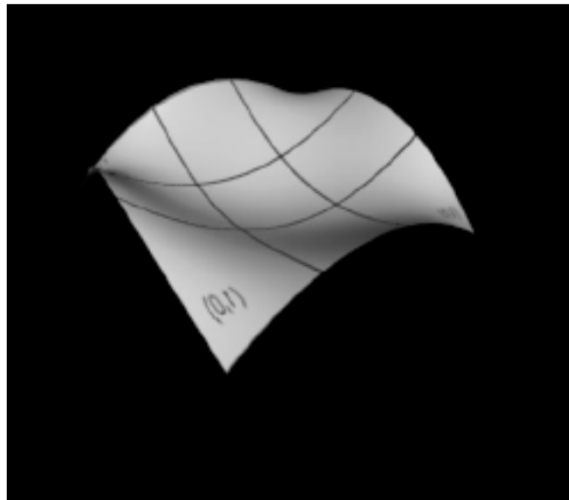
Curve Representation

▸ Specify many points along a curve, connect with lines?

  ▸ Difficult to get precise, smooth results across magnification levels
  ▸ Large storage and CPU requirements
  ▸ How many points are enough?

▸ Specify a curve using a small number of "control points"

  ▸ Known as a *spline curve* or just *spline*

Control point

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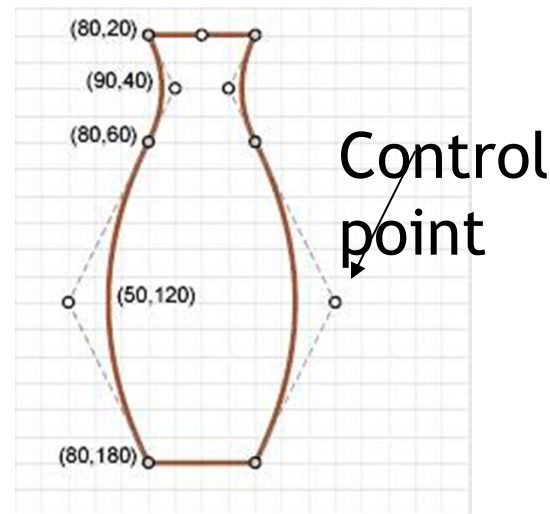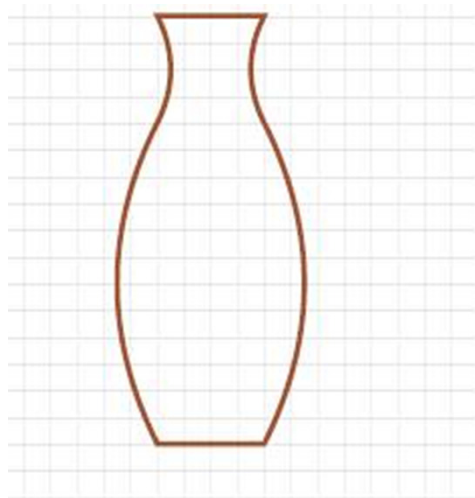
Spline: Definition

▸ Wikipedia:

 ▸ Term comes from flexible spline devices used by shipbuilders and draftsmen to draw smooth shapes.

 ▸ Spline consists of a long strip fixed in position at a number of points that relaxes to form a smooth curve passing through those points.
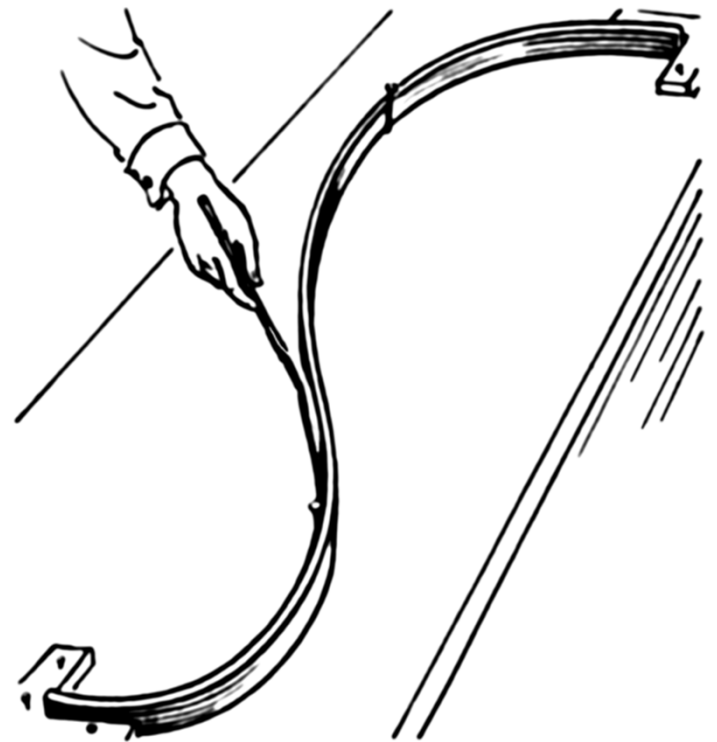
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Interpolating Control Points

▸ "Interpolating" means that curve goes through all control points

▸ A.k.a. "Anchor Points"

▸ Seems most intuitive

▸ Surprisingly, not usually the best choice

  ▸ Hard to predict behavior

  ▸ Hard to get aesthetically pleasing curves

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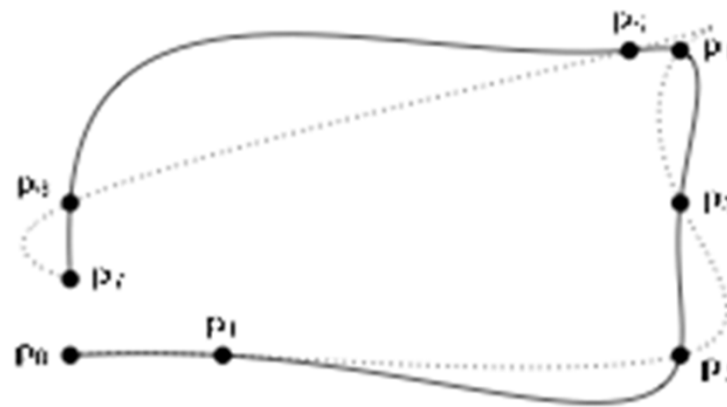
Approximating Control Points

▶ Curve is "influenced" by control points
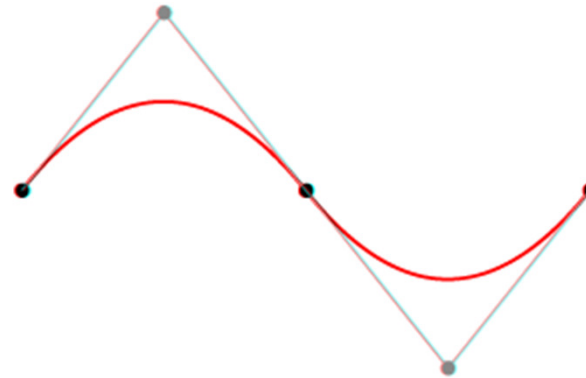
▶ Various types

▶ Most common: polynomial functions

  ▶ Bézier spline (our focus)

  ▶ B-spline (generalization of Bézier spline)

  ▶ NURBS (Non Uniform Rational Basis Spline): used in CAD tools

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Mathematical Definition

- A vector valued function of one variable $\mathbf{x}(t)$
  - Given t, compute a 3D point $\mathbf{x}=(x,y,z)$
  - Could be interpreted as three functions: $x(t)$, $y(t)$, $z(t)$
  - Parameter t "moves a point along the curve"

$\mathbf{x}(t)$

$\mathbf{x}(0.0)$ $\mathbf{x}(0.5)$ $\mathbf{x}(1.0)$

z

y

x

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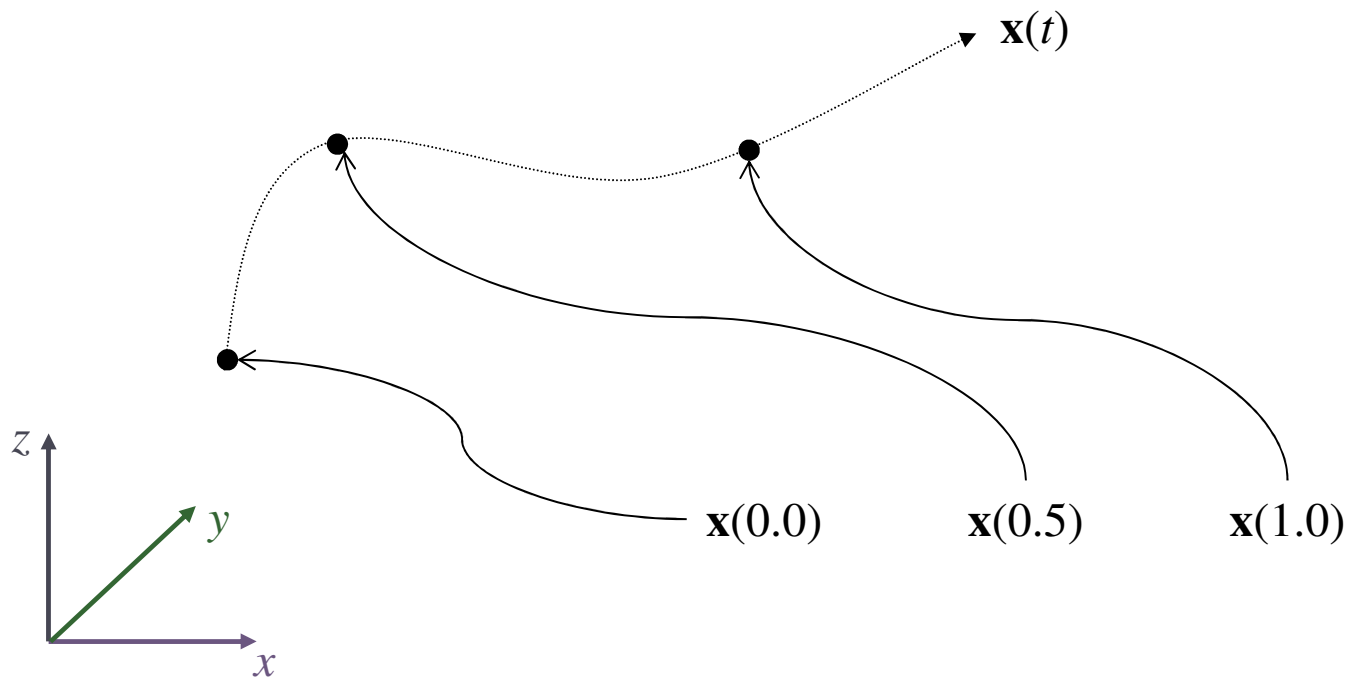
Tangent Vector

▸ Derivative $\mathbf{x}'(t) = \dfrac{d\mathbf{x}}{dt} = (x'(t), y'(t), z'(t))$

▸ Vector x' points in direction of movement

▸ Length corresponds to speed
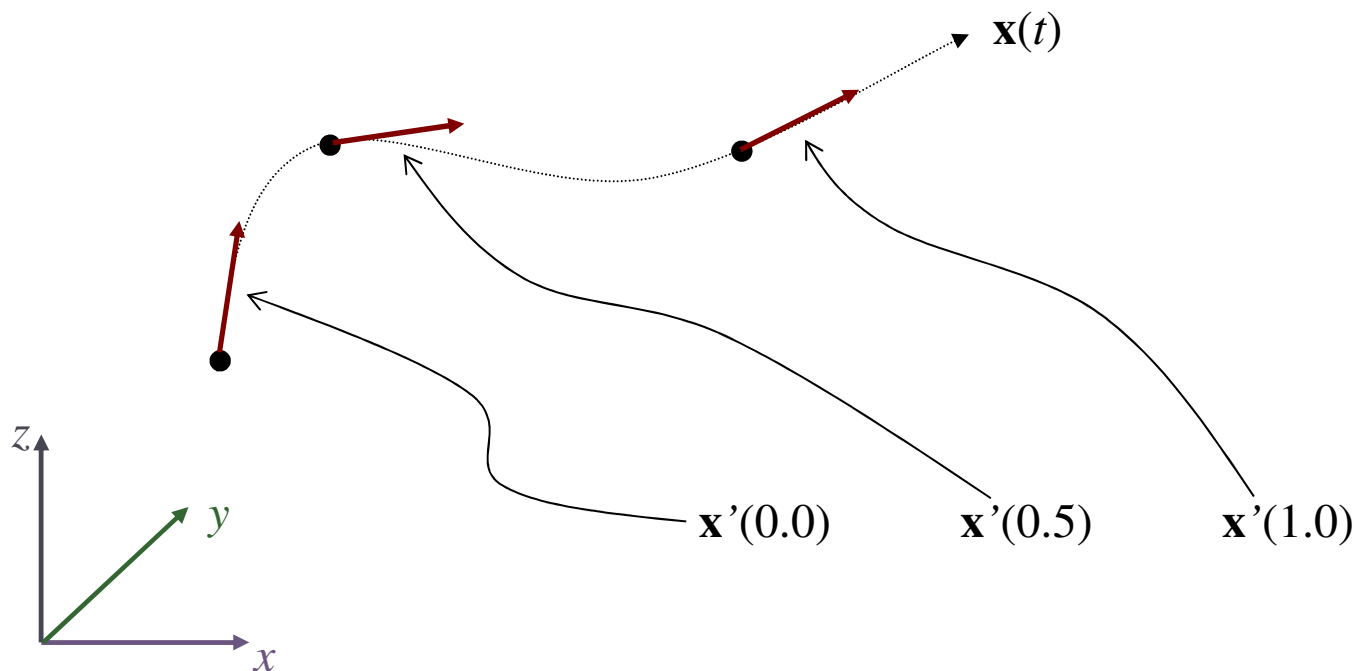
$\mathbf{x}(t)$

z

y

x

$\mathbf{x}'(0.0)$ $\mathbf{x}'(0.5)$ $\mathbf{x}'(1.0)$

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Polynomial Functions

▸ **Linear:** $f(t) = at + b$
(1st order)
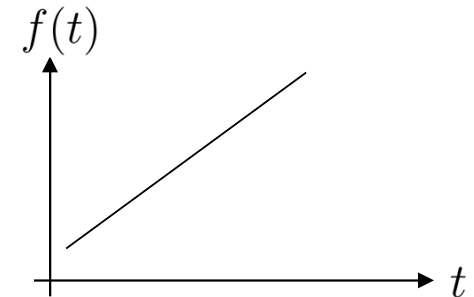
▸ **Quadratic:** $f(t) = at^2 + bt + c$
(2nd order)
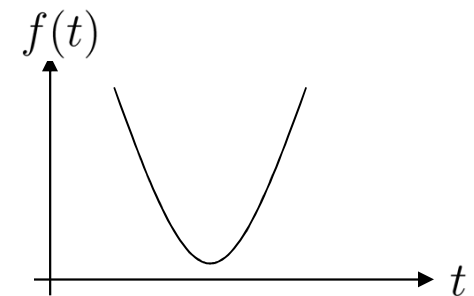
▸ **Cubic:** $f(t) = at^3 + bt^2 + ct + d$
(3rd order)

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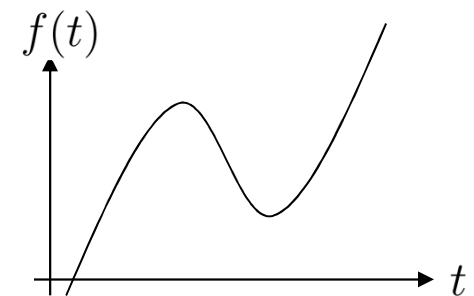
Polynomial Curves

- Linear $\mathbf{x}(t) = \mathbf{a}t + \mathbf{b}$

$$\mathbf{x} = (x, y, z), \mathbf{a} = (a_x, a_y, a_z), \mathbf{b} = (b_x, b_y, b_z)$$

- Evaluated as:

$$x(t) = a_x t + b_x$$
$$y(t) = a_y t + b_y$$
$$z(t) = a_z t + b_z$$

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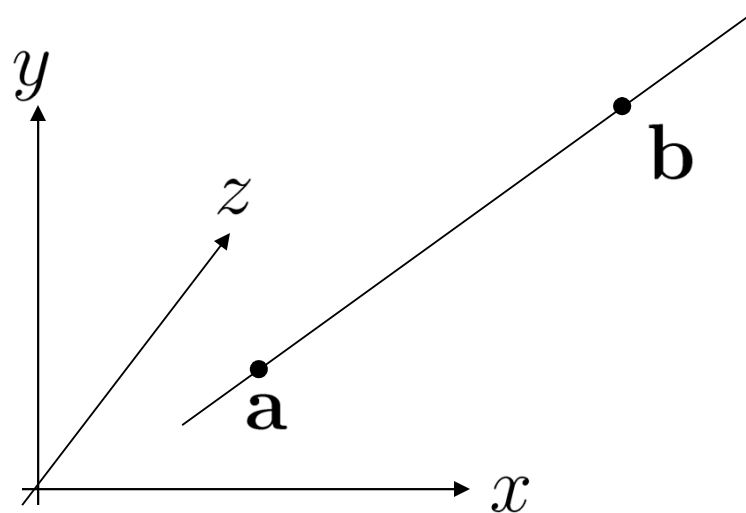
Polynomial Curves

▶ **Quadratic:** $\mathbf{x}(t) = \mathbf{a}t^2 + \mathbf{b}t + \mathbf{c}$
(2nd order)
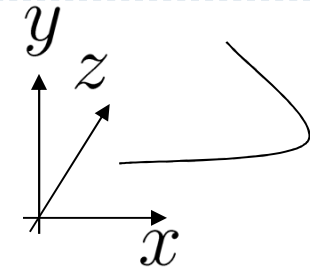
▶ **Cubic:** $\mathbf{x}(t) = \mathbf{a}t^3 + \mathbf{b}t^2 + \mathbf{c}t + \mathbf{d}$
(3rd order)
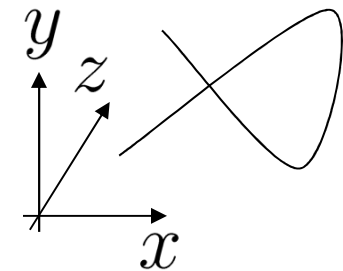
▶ We usually define the curve for $0 \leq t \leq 1$

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Control Points

- Polynomial coefficients **a, b, c, d** can be interpreted as *control points*
  - Remember: **a, b, c, d** have x, y, z components each
- Unfortunately, they do not intuitively describe the shape of the curve
- Goal: intuitive control points

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Control Points

▶ How many control points?

  ▶ Two points define a line (1st order)

  ▶ Three points define a quadratic curve (2nd order)

  ▶ Four points define a cubic curve (3rd order)

  ▶ $k+1$ points define a k-order curve

▶ Let's start with a line…

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First Order Curve

- Based on linear interpolation (LERP)
  - Weighted average between two values
  - "Value" could be a number, vector, color, …
- Interpolate between points $\mathbf{p}_0$ and $\mathbf{p}_1$ with parameter t
  - Defines a "curve" that is straight (first-order spline)
  - $t=0$ corresponds to $\mathbf{p}_0$
  - $t=1$ corresponds to $\mathbf{p}_1$
  - $t=0.5$ corresponds to midpoint

$\mathbf{p}_1$

$t=1$

$\mathbf{p}_0$

$0 < t < 1$

$t=0$
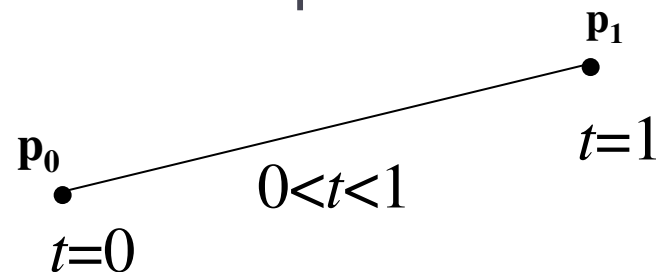
$$\mathbf{x}(t) = Lerp\left(t, \mathbf{p}_0, \mathbf{p}_1\right) = (1-t)\mathbf{p}_0 + t\ \mathbf{p}_1$$

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Linear Interpolation

▶ **Three equivalent ways to write it**

  ▶ Expose different properties

1. **Regroup for points p**

$$\mathbf{x}(t) = \mathbf{p}_0(1 - t) + \mathbf{p}_1 t$$

2. **Regroup for** t

$$\mathbf{x}(t) = (\mathbf{p}_1 - \mathbf{p}_0)t + \mathbf{p}_0$$

3. **Matrix form**

$$\mathbf{x}(t) = \begin{bmatrix} \mathbf{p}_0 & \mathbf{p}_1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} t \\ 1 \end{bmatrix}$$

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Weighted Average

$$\mathbf{x}(t) = (1 - t)\mathbf{p}_0 + (t)\mathbf{p}_1$$

$$= B_0(t)\,\mathbf{p}_0 + B_1(t)\mathbf{p}_1, \text{ where } B_0(t) = 1 - t \text{ and } B_1(t) = t$$

▸ Weights are a function of t

   ▸ Sum is always 1, for any value of t

   ▸ Also known as *blending functions*

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Linear Polynomial

$$\mathbf{x}(t) = \underbrace{(\mathbf{p}_1 - \mathbf{p}_0)}_{\substack{\text{vector} \\ \mathbf{a}}} t + \underbrace{\mathbf{p}_0}_{\substack{\text{point} \\ \mathbf{b}}}$$

▸ Curve is based at point $\mathbf{p}_0$

▸ Add the vector, scaled by t

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Matrix Form

$$\mathbf{x}(t) = \begin{bmatrix} \mathbf{p}_0 & \mathbf{p}_1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} t \\ 1 \end{bmatrix} = \mathbf{GBT}$$
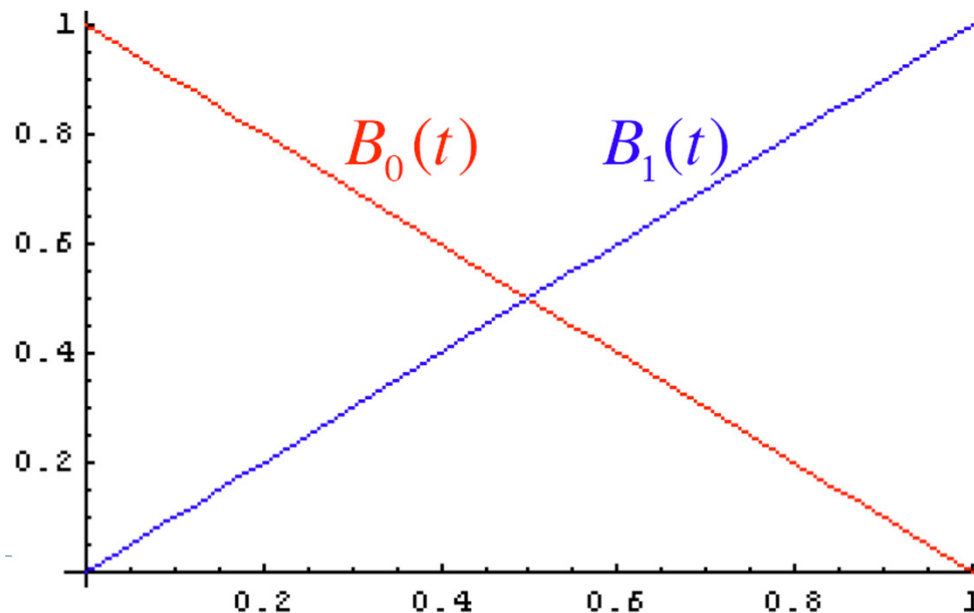
▸ Geometry matrix $\quad \mathbf{G} = \begin{bmatrix} \mathbf{p}_0 & \mathbf{p}_1 \end{bmatrix}$

▸ Geometric basis $\quad \mathbf{B} = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$

▸ Polynomial basis $\quad T = \begin{bmatrix} t \\ 1 \end{bmatrix}$

▸ In components

$$\mathbf{x}(t) = \begin{bmatrix} p_{0x} & p_{1x} \\ p_{0y} & p_{1y} \\ p_{0z} & p_{1z} \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} t \\ 1 \end{bmatrix}$$

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Matrix Form

$$\mathbf{x}(t) = \begin{bmatrix} \mathbf{p}_0 & \mathbf{p}_1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} t \\ 1 \end{bmatrix} = \mathbf{GBT}$$

▸ Geometry matrix $\quad \mathbf{G} = \begin{bmatrix} \mathbf{p}_0 & \mathbf{p}_1 \end{bmatrix}$

▸ Geometric basis $\quad \mathbf{B} = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$

▸ Polynomial basis

$$T = \begin{bmatrix} t \\ 1 \end{bmatrix}$$

▸ In components

$$\mathbf{x}(t) = \begin{bmatrix} p_{0x} & p_{1x} \\ p_{0y} & p_{1y} \\ p_{0z} & p_{1z} \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} t \\ 1 \end{bmatrix}$$

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Tangent

- For a straight line, the tangent is constant

$$\mathbf{x}'(t) = \mathbf{p}_1 - \mathbf{p}_0$$

- Weighted average $\quad \mathbf{x}'(t) = (-1)\mathbf{p}_0 + (+1)\mathbf{p}_1$

- Polynomial $\quad\quad \mathbf{x}'(t) = 0t + (\mathbf{p}_1 - \mathbf{p}_0)$

- Matrix form $\quad \mathbf{x}'(t) = \begin{bmatrix} \mathbf{p}_0 & \mathbf{p}_1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

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