CSE 167: Introduction to Computer Graphics Lecture #12: Curves

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Announcements

- Homework 6 due Friday at 1pm
- Next Monday: Midterm review
- Midterm #2 on Thu May 20th

Lecture Overview

- Polynomial Curves
  - Introduction
  - Polynomial functions
- Bézier Curves
  - Introduction
  - Drawing Bézier curves
  - Piecewise Bézier curves

Linear Interpolation

- Three equivalent ways to write it
  - Expose different properties

1. Regroup for points **p**

$$\mathbf{x}(t) = \mathbf{p}_0(1-t) + \mathbf{p}_1 t$$

2. Regroup for t

$$\mathbf{x}(t) = (\mathbf{p}_1 - \mathbf{p}_0)t + \mathbf{p}_0$$

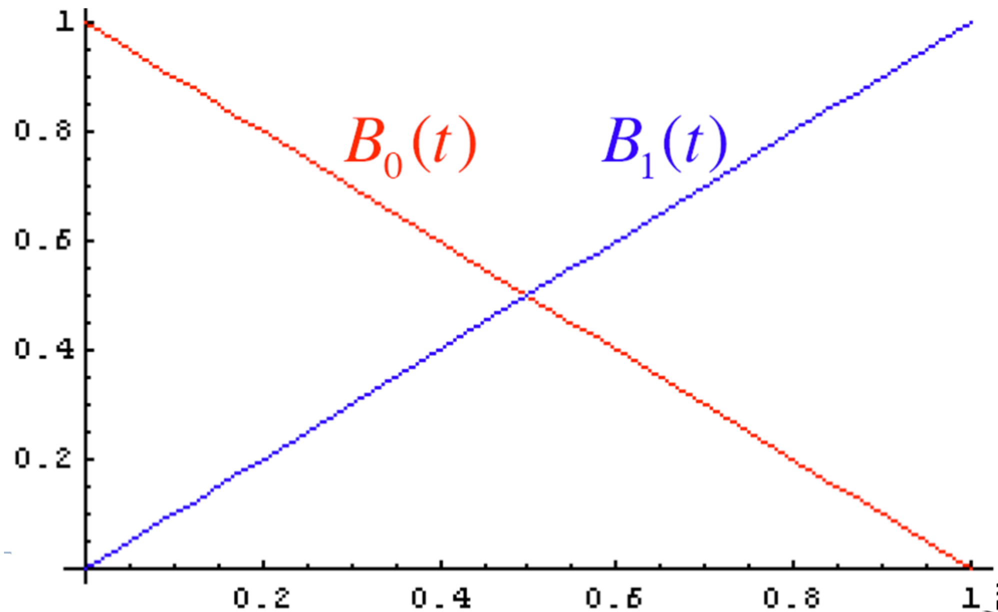
3. Matrix form

$$\mathbf{x}(t) = \begin{bmatrix} \mathbf{p}_0 & \mathbf{p}_1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} t \\ 1 \end{bmatrix}$$

Weighted Average

$$\mathbf{x}(t) = (1-t)\mathbf{p}_0 + \quad (t)\mathbf{p}_1$$
$$= B_0(t)\,\mathbf{p}_0 + B_1(t)\mathbf{p}_1, \text{ where } B_0(t) = 1-t \text{ and } B_1(t) = t$$

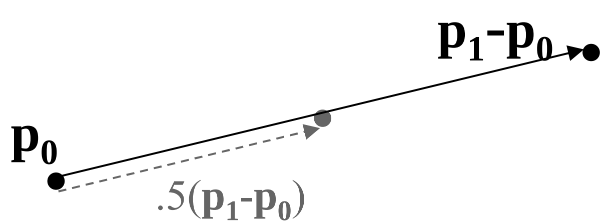
- Weights are a function of t
  - Sum is always 1, for any value of t
  - Also known as *blending functions*



Linear Polynomial

$$\mathbf{x}(t) = \underbrace{(\mathbf{p}_1 - \mathbf{p}_0)}_{\substack{\text{vector} \\ \mathbf{a}}} t + \underbrace{\mathbf{p}_0}_{\substack{\text{point} \\ \mathbf{b}}}$$

- Curve is based at point $\mathbf{p_0}$
- Add the vector, scaled by t



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Matrix Form

$$\mathbf{x}(t) = \begin{bmatrix} \mathbf{p}_0 & \mathbf{p}_1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} t \\ 1 \end{bmatrix} = \mathbf{GBT}$$

- Geometry matrix $\mathbf{G} = \begin{bmatrix} \mathbf{p}_0 & \mathbf{p}_1 \end{bmatrix}$
- Geometric basis $\mathbf{B} = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$
- Polynomial basis $T = \begin{bmatrix} t \\ 1 \end{bmatrix}$
- In components

$$\mathbf{x}(t) = \begin{bmatrix} p_{0x} & p_{1x} \\ p_{0y} & p_{1y} \\ p_{0z} & p_{1z} \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} t \\ 1 \end{bmatrix}$$

Matrix Form

$$\mathbf{x}(t) = \begin{bmatrix} \mathbf{p}_0 & \mathbf{p}_1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} t \\ 1 \end{bmatrix} = \mathbf{GBT}$$

- Geometry matrix $\mathbf{G} = \begin{bmatrix} \mathbf{p}_0 & \mathbf{p}_1 \end{bmatrix}$
- Geometric basis $\mathbf{B} = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$
- Polynomial basis $T = \begin{bmatrix} t \\ 1 \end{bmatrix}$
- In components

$$\mathbf{x}(t) = \begin{bmatrix} p_{0x} & p_{1x} \\ p_{0y} & p_{1y} \\ p_{0z} & p_{1z} \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} t \\ 1 \end{bmatrix}$$

Tangent

- For a straight line, the tangent is constant

$$\mathbf{x}'(t) = \mathbf{p}_1 - \mathbf{p}_0$$

- Weighted average $\mathbf{x}'(t) = (-1)\mathbf{p}_0 + (+1)\mathbf{p}_1$

- Polynomial $\mathbf{x}'(t) = 0t + (\mathbf{p}_1 - \mathbf{p}_0)$

- Matrix form $\mathbf{x}'(t) = \begin{bmatrix} \mathbf{p}_0 & \mathbf{p}_1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

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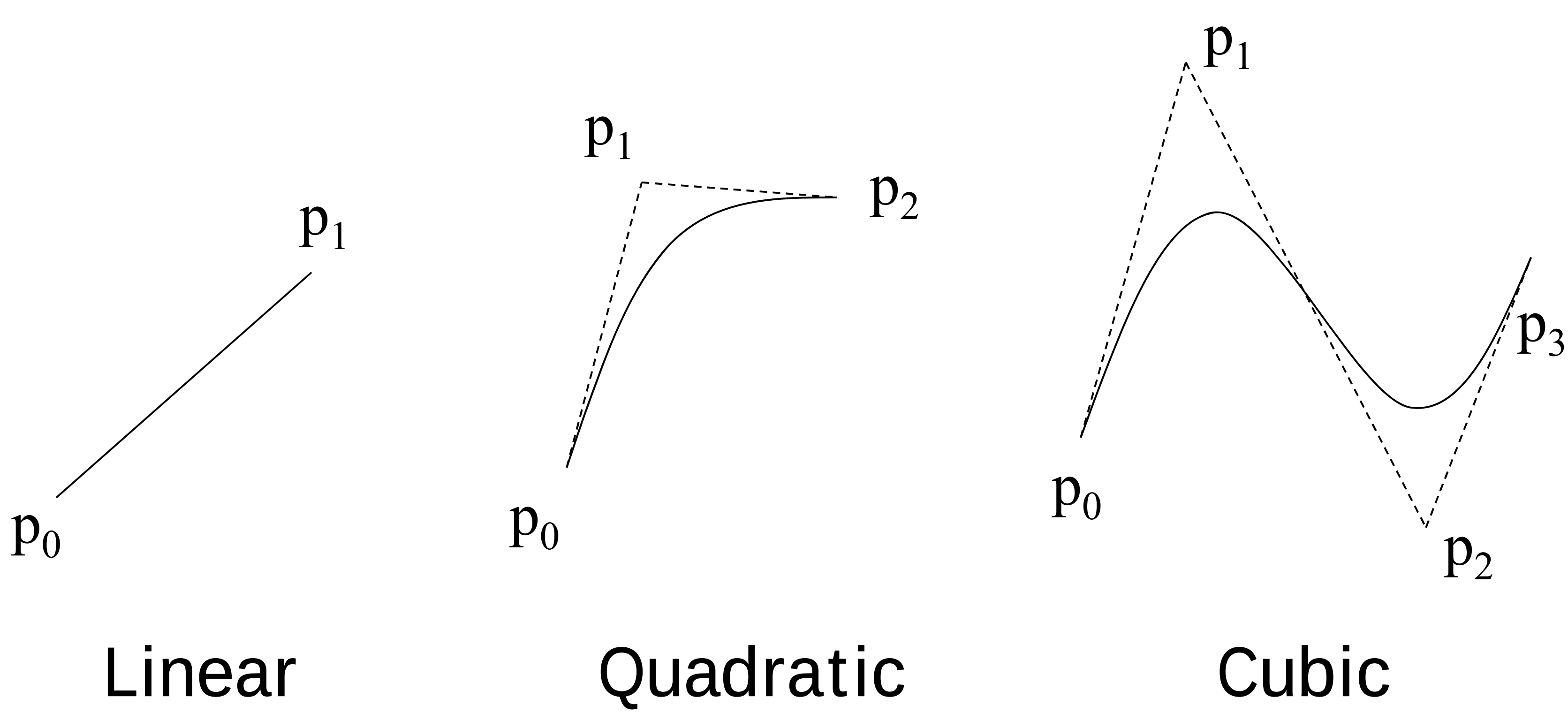
Bézier Curves

- Are a higher order extension of linear interpolation



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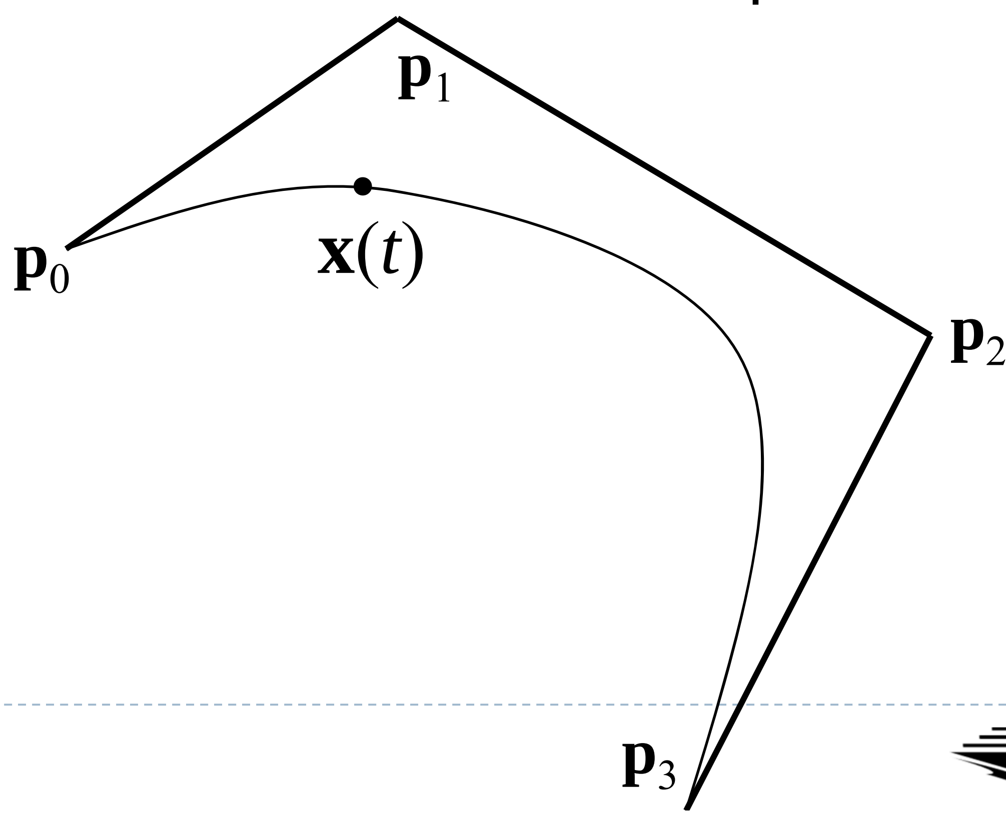
Bézier Curves

- Give intuitive control over curve with control points
  - Endpoints are interpolated, intermediate points are approximated
  - Convex Hull property
- Many demo applets online, for example:
  - Demo: http://www.cs.princeton.edu/~min/cs426/jar/bezier.html
  - http://www.theparticle.com/applets/nyu/BezierApplet/
  - http://www.sunsite.ubc.ca/LivingMathematics/V001N01/UBCExamples/Bezier/bezier.html

Cubic Bézier Curve

- Most commonly used case
- Defined by four control points:
  - Two interpolated endpoints (points are on the curve)
  - Two points control the tangents at the endpoints
- Points **x** on curve defined as function of parameter t



Algorithmic Construction

- Algorithmic construction
  - *De Casteljau* algorithm, developed at Citroen in 1959, named after its inventor Paul de Casteljau (pronounced "Cast-all-'Joe")
  - Developed independently from Bézier's work:
    Bézier created the formulation using blending functions, Casteljau devised the recursive interpolation algorithm

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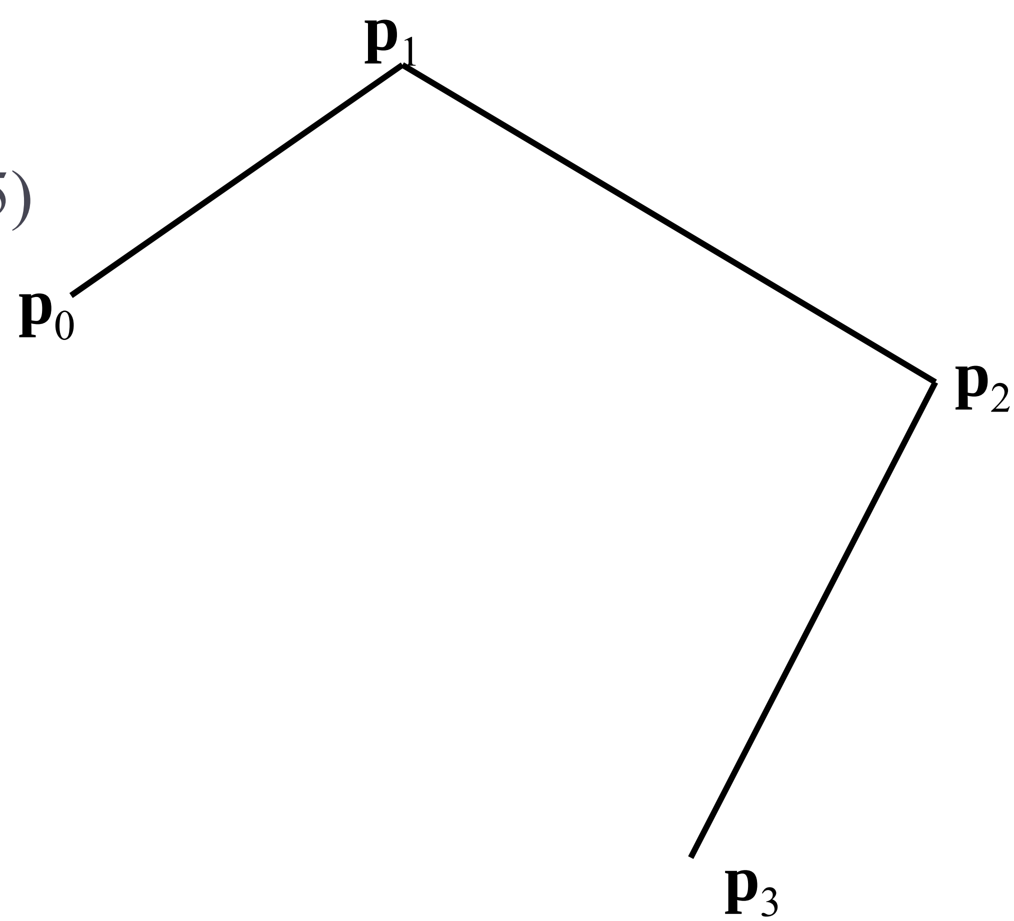
De Casteljau Algorithm

- A recursive series of linear interpolations
  - Works for any order Bezier function, not only cubic
- Not very efficient to evaluate
  - Other forms more commonly used
- But:
  - Gives intuition about the geometry
  - Useful for subdivision

De Casteljau Algorithm

- Given:
  - Four control points
  - A value of t (here $t \approx 0.25$)

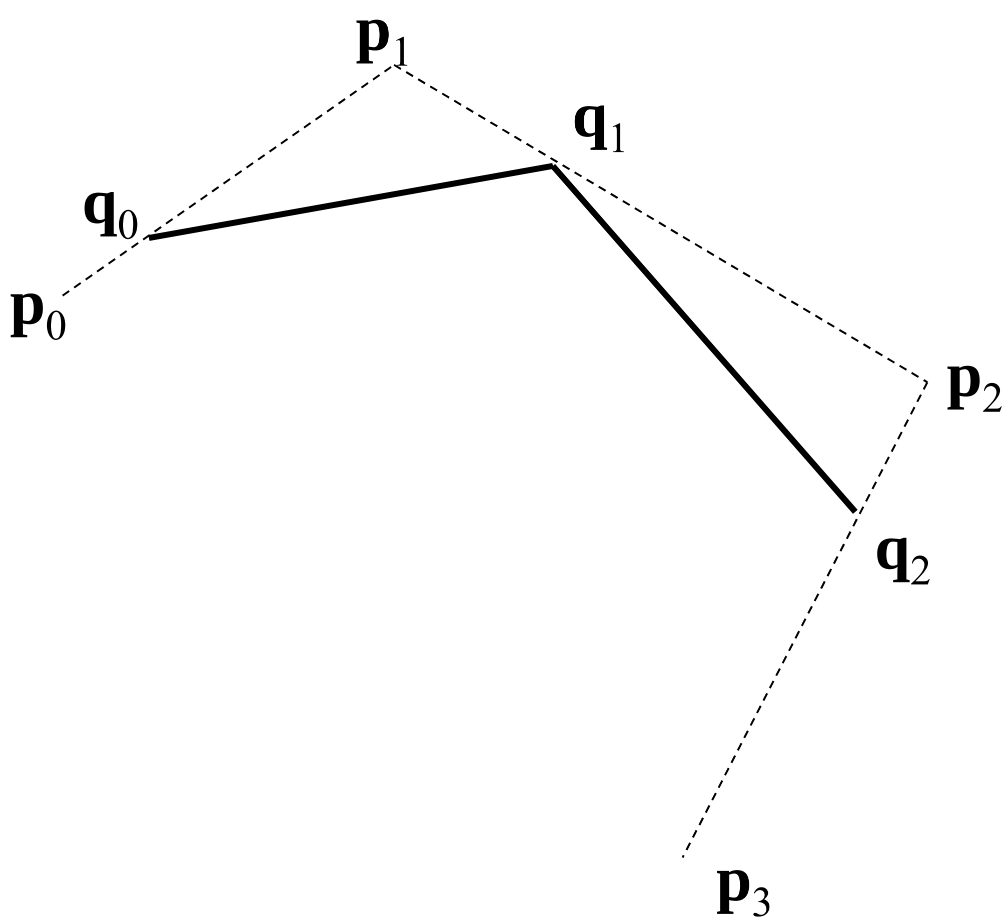


De Casteljau Algorithm

$$\mathbf{q}_0(t) = Lerp(t, \mathbf{p}_0, \mathbf{p}_1)$$

$$\mathbf{q}_1(t) = Lerp(t, \mathbf{p}_1, \mathbf{p}_2)$$

$$\mathbf{q}_2(t) = Lerp(t, \mathbf{p}_2, \mathbf{p}_3)$$



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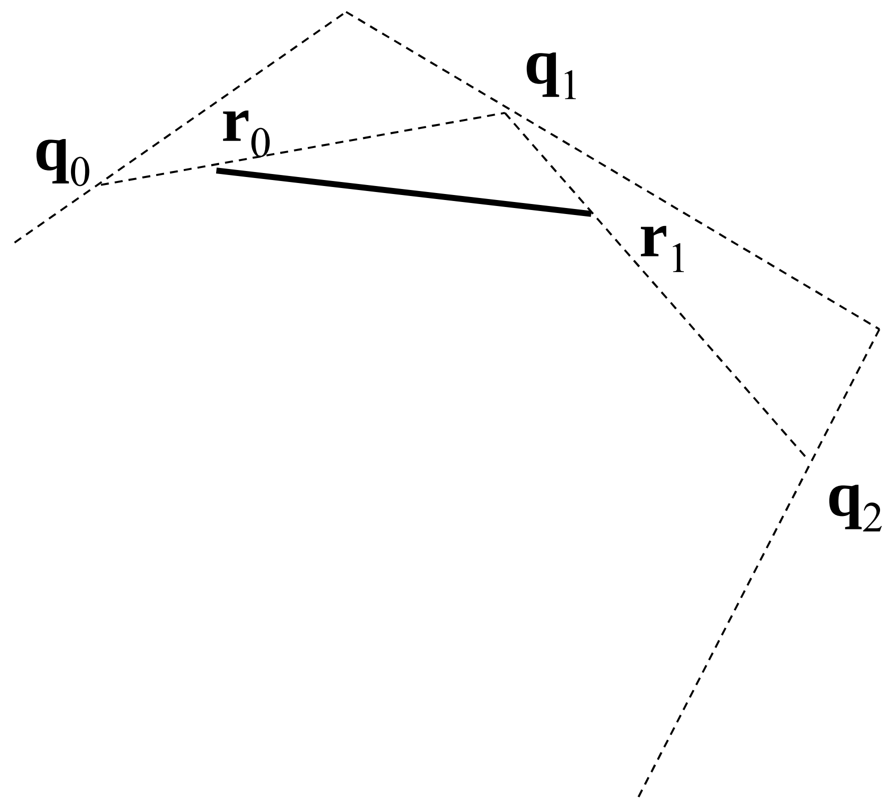
De Casteljau Algorithm

$$\mathbf{r}_0(t) = Lerp\left(t, \mathbf{q}_0(t), \mathbf{q}_1(t)\right)$$

$$\mathbf{r}_1(t) = Lerp\left(t, \mathbf{q}_1(t), \mathbf{q}_2(t)\right)$$



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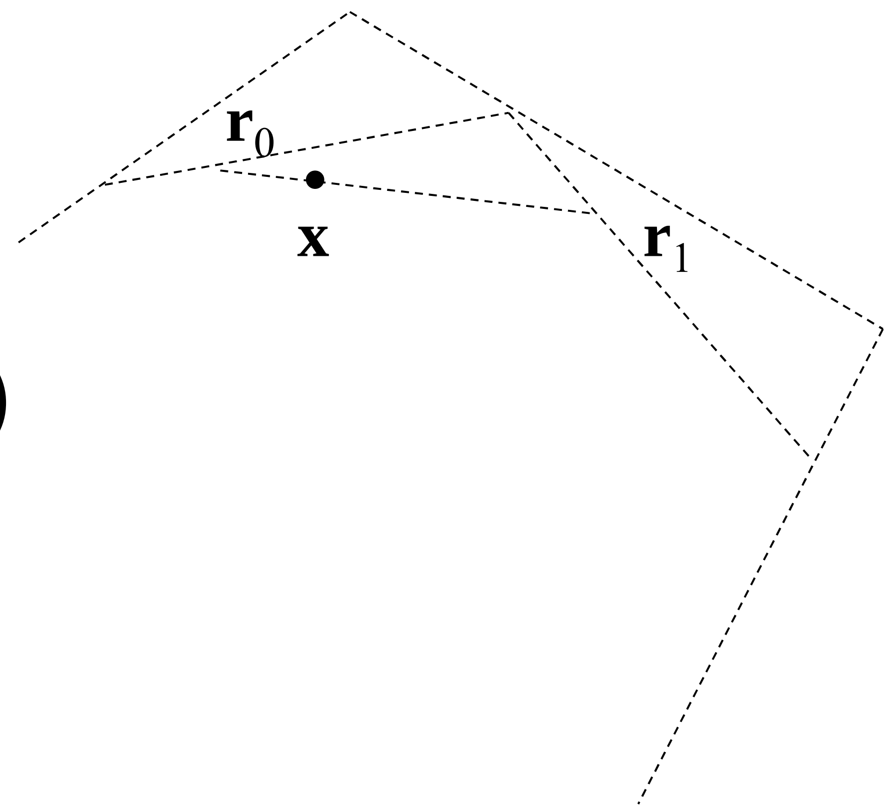
De Casteljau Algorithm



$$\mathbf{x}(t) = Lerp\left(t, \mathbf{r}_0(t), \mathbf{r}_1(t)\right)$$

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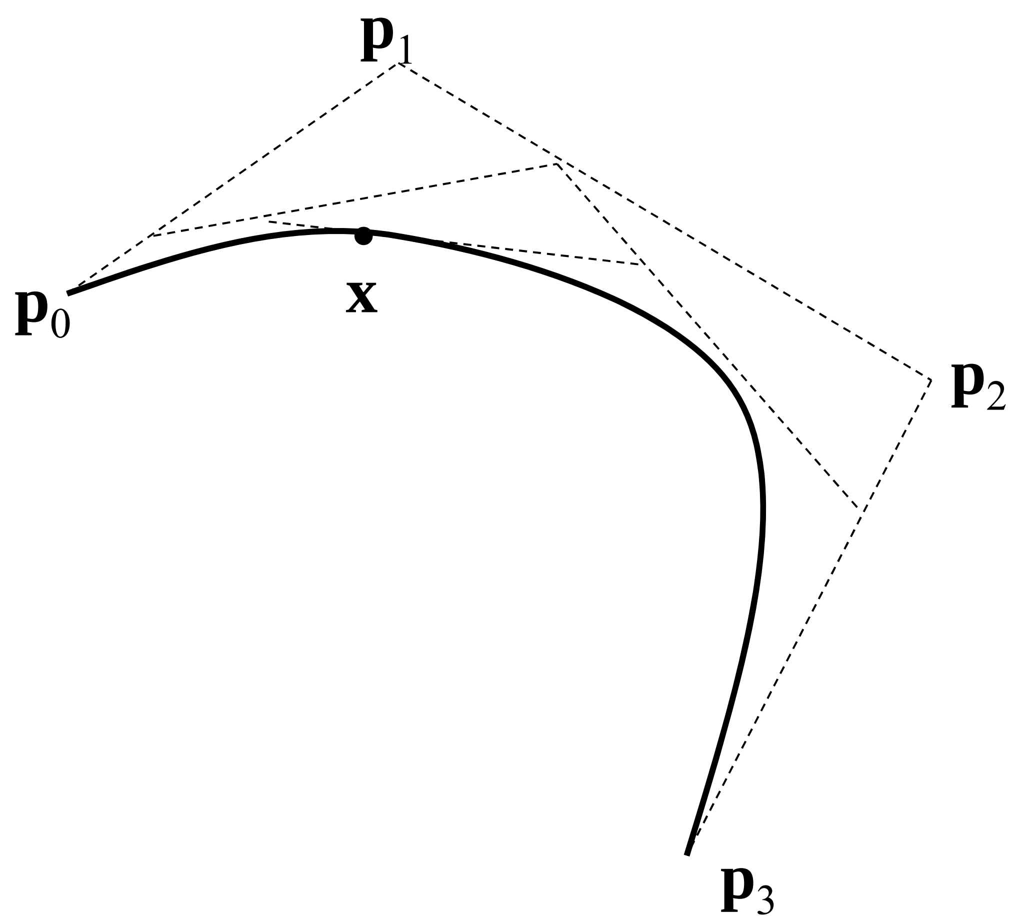
De Casteljau Algorithm



- Applets
  - Demo: http://www2.mat.dtu.dk/people/J.Gravesen/cagd/decast.html
  - http://www.caffeineowl.com/graphics/2d/vectorial/bezierintro.html

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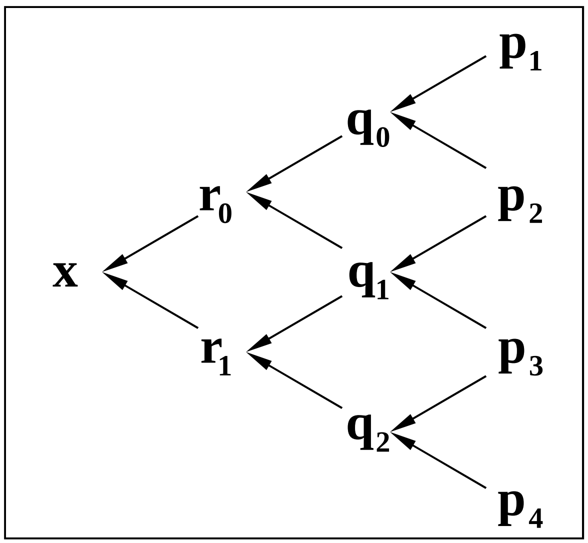
Recursive Linear Interpolation

$$\mathbf{x} = Lerp\left(t, \mathbf{r}_0, \mathbf{r}_1\right) \quad \begin{array}{l} \mathbf{r}_0 = Lerp\left(t, \mathbf{q}_0, \mathbf{q}_1\right) \\ \mathbf{r}_1 = Lerp\left(t, \mathbf{q}_1, \mathbf{q}_2\right) \end{array} \quad \begin{array}{l} \mathbf{q}_0 = Lerp\left(t, \mathbf{p}_0, \mathbf{p}_1\right) \\ \mathbf{q}_1 = Lerp\left(t, \mathbf{p}_1, \mathbf{p}_2\right) \\ \mathbf{q}_2 = Lerp\left(t, \mathbf{p}_2, \mathbf{p}_3\right) \end{array} \quad \begin{array}{l} \mathbf{p}_0 \\ \mathbf{p}_1 \\ \mathbf{p}_2 \\ \mathbf{p}_3 \end{array}$$



Expand the LERPs

$$\mathbf{q}_0(t) = Lerp\left(t, \mathbf{p}_0, \mathbf{p}_1\right) = (1-t)\mathbf{p}_0 + t\mathbf{p}_1$$

$$\mathbf{q}_1(t) = Lerp\left(t, \mathbf{p}_1, \mathbf{p}_2\right) = (1-t)\mathbf{p}_1 + t\mathbf{p}_2$$

$$\mathbf{q}_2(t) = Lerp\left(t, \mathbf{p}_2, \mathbf{p}_3\right) = (1-t)\mathbf{p}_2 + t\mathbf{p}_3$$

$$\mathbf{r}_0(t) = Lerp\left(t, \mathbf{q}_0(t), \mathbf{q}_1(t)\right) = (1-t)\left((1-t)\mathbf{p}_0 + t\mathbf{p}_1\right) + t\left((1-t)\mathbf{p}_1 + t\mathbf{p}_2\right)$$

$$\mathbf{r}_1(t) = Lerp\left(t, \mathbf{q}_1(t), \mathbf{q}_2(t)\right) = (1-t)\left((1-t)\mathbf{p}_1 + t\mathbf{p}_2\right) + t\left((1-t)\mathbf{p}_2 + t\mathbf{p}_3\right)$$

$$\begin{aligned} \mathbf{x}(t) &= Lerp\left(t, \mathbf{r}_0(t), \mathbf{r}_1(t)\right) \\ &= (1-t)\left((1-t)\left((1-t)\mathbf{p}_0 + t\mathbf{p}_1\right) + t\left((1-t)\mathbf{p}_1 + t\mathbf{p}_2\right)\right) \\ &\quad + t\left((1-t)\left((1-t)\mathbf{p}_1 + t\mathbf{p}_2\right) + t\left((1-t)\mathbf{p}_2 + t\mathbf{p}_3\right)\right) \end{aligned}$$

Weighted Average of Control Points

- Regroup for p:

$$\mathbf{x}(t) = (1-t)\Big((1-t)\big((1-t)\mathbf{p}_0 + t\mathbf{p}_1\big) + t\big((1-t)\mathbf{p}_1 + t\mathbf{p}_2\big)\Big)$$
$$+t\Big((1-t)\big((1-t)\mathbf{p}_1 + t\mathbf{p}_2\big) + t\big((1-t)\mathbf{p}_2 + t\mathbf{p}_3\big)\Big)$$

$$\mathbf{x}(t) = (1-t)^3\,\mathbf{p}_0 + 3(1-t)^2\,t\mathbf{p}_1 + 3(1-t)t^2\mathbf{p}_2 + t^3\mathbf{p}_3$$

$$\mathbf{x}(t) = \overbrace{\left(-t^3 + 3t^2 - 3t + 1\right)}^{B_0(t)}\mathbf{p}_0 + \overbrace{\left(3t^3 - 6t^2 + 3t\right)}^{B_1(t)}\mathbf{p}_1$$
$$+\underbrace{\left(-3t^3 + 3t^2\right)}_{B_2(t)}\mathbf{p}_2 + \underbrace{\left(t^3\right)}_{B_3(t)}\mathbf{p}_3$$

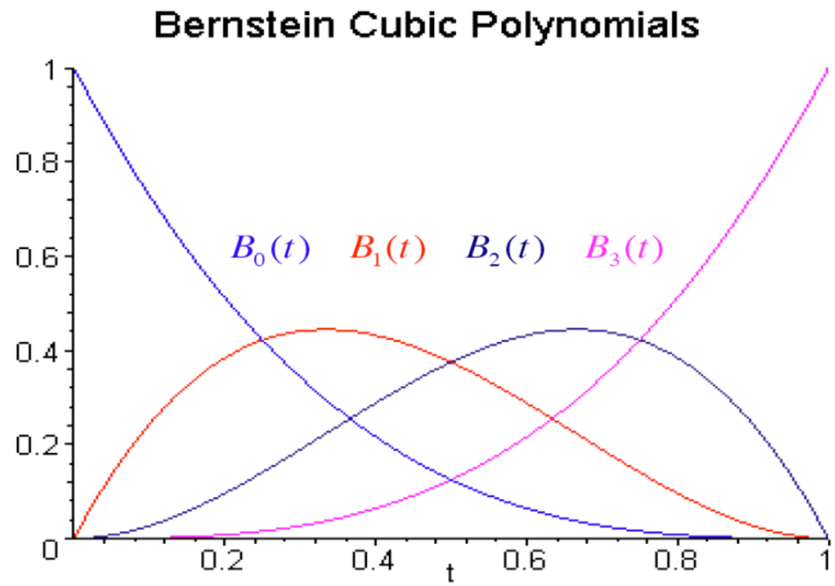
Cubic Bernstein Polynomials

$$\mathbf{x}(t) = B_0(t)\mathbf{p}_0 + B_1(t)\mathbf{p}_1 + B_2(t)\mathbf{p}_2 + B_3(t)\mathbf{p}_3$$

The cubic *Bernstein polynomials :*

$$B_0(t) = -t^3 + 3t^2 - 3t + 1$$
$$B_1(t) = 3t^3 - 6t^2 + 3t$$
$$B_2(t) = -3t^3 + 3t^2$$
$$B_3(t) = t^3$$

$$\sum B_i(t) = 1$$



- Weights $B_i(t)$ add up to 1 for any value of *t*

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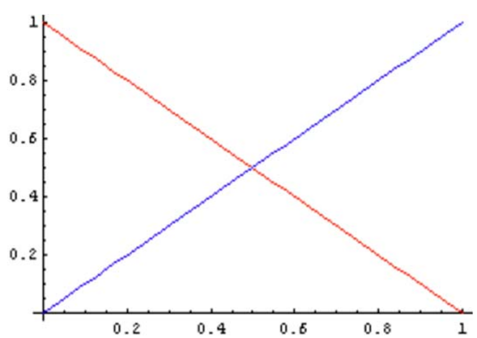
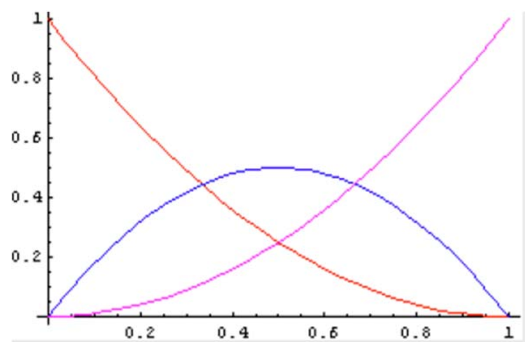
General Bernstein Polynomials

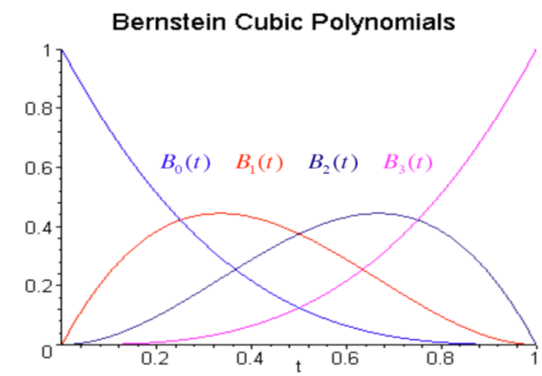
$$B_0^1(t) = -t + 1$$
$$B_1^1(t) = t$$

$$B_0^2(t) = t^2 - 2t + 1$$
$$B_1^2(t) = -2t^2 + 2t$$
$$B_2^2(t) = t^2$$

$$B_0^3(t) = -t^3 + 3t^2 - 3t + 1$$
$$B_1^3(t) = 3t^3 - 6t^2 + 3t$$
$$B_2^3(t) = -3t^3 + 3t^2$$
$$B_3^3(t) = t^3$$







$$B_i^n(t) = \binom{n}{i}(1-t)^{n-i}(t)^i$$

$$\binom{n}{i} = \frac{n!}{i!(n-i)!}$$

$$\sum B_i^n(t) = 1$$

n! = factorial of n
(n+1)! = n! x (n+1)

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General Bézier Curves

- *n*th-order Bernstein polynomials form *n*th-order Bézier curves

$$B_i^n(t) = \binom{n}{i}(1-t)^{n-i}(t)^i$$

$$\mathbf{x}(t) = \sum_{i=0}^{n} B_i^n(t)\mathbf{p}_i$$

Bézier Curve Properties

Overview:

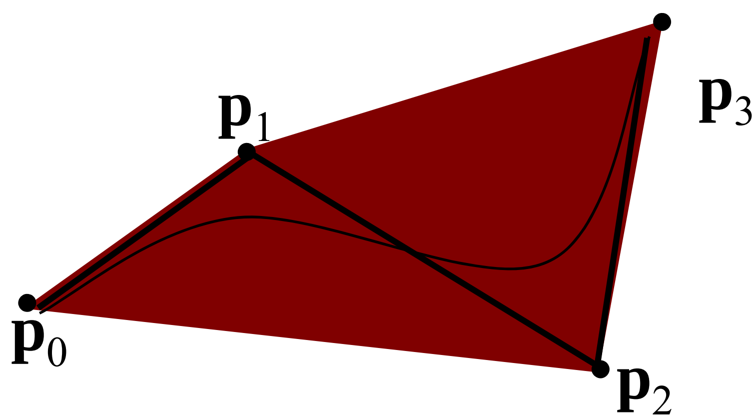
- Convex Hull property
- Affine Invariance

Definitions

- Convex hull of a set of points:
  - Polyhedral volume created such that all lines connecting any two points lie completely inside it (or on its boundary)
- Convex combination of a set of points:
  - Weighted average of the points, where all weights between 0 and 1, sum up to 1
- Any convex combination of a set of points lies within the convex hull

Convex Hull Property

- A Bézier curve is a convex combination of the control points (by definition, see Bernstein polynomials)
- A Bézier curve is always inside the convex hull
  - Makes curve predictable
  - Allows culling, intersection testing, adaptive tessellation
- Demo: http://www.cs.princeton.edu/~min/cs426/jar/bezier.html



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