CSE 167:
Introduction to Computer Graphics
Lecture #12: Surface Patches

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Announcements

- Tomorrow: grading project 4
- Monday: midterm discussion
- Next Thursday: midterm #2

Overview

- Bi-linear patch
- Bi-cubic Bézier patch

Curved Surfaces

Curves

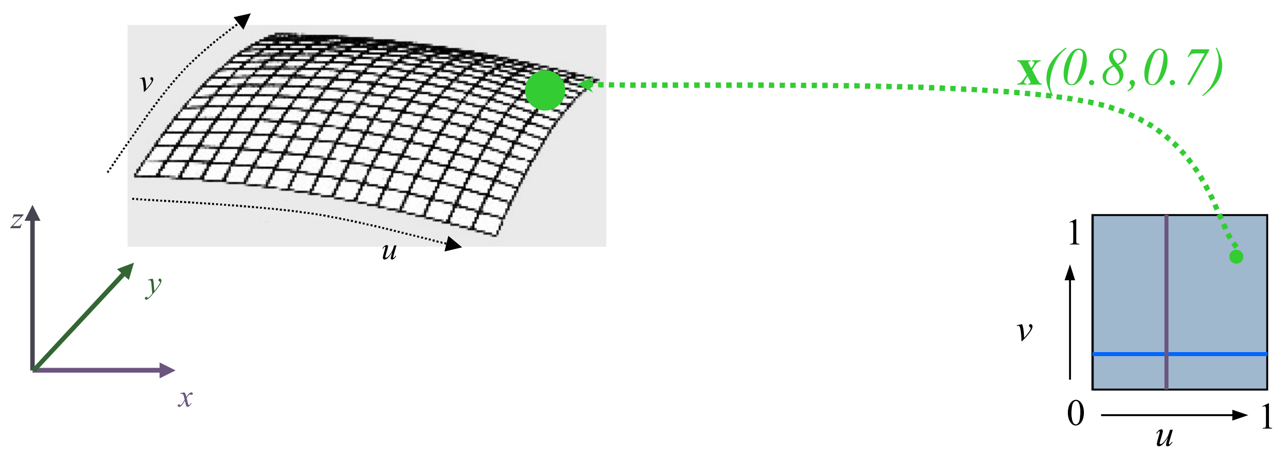
- Described by a 1D series of control points
- A function $\mathbf{x}(t)$
- Segments joined together to form a longer curve

Surfaces

- Described by a 2D mesh of control points
- Parameters have two dimensions (two dimensional parameter domain)
- A function $\mathbf{x}(u,v)$
- Patches joined together to form a bigger surface

Parametric Surface Patch

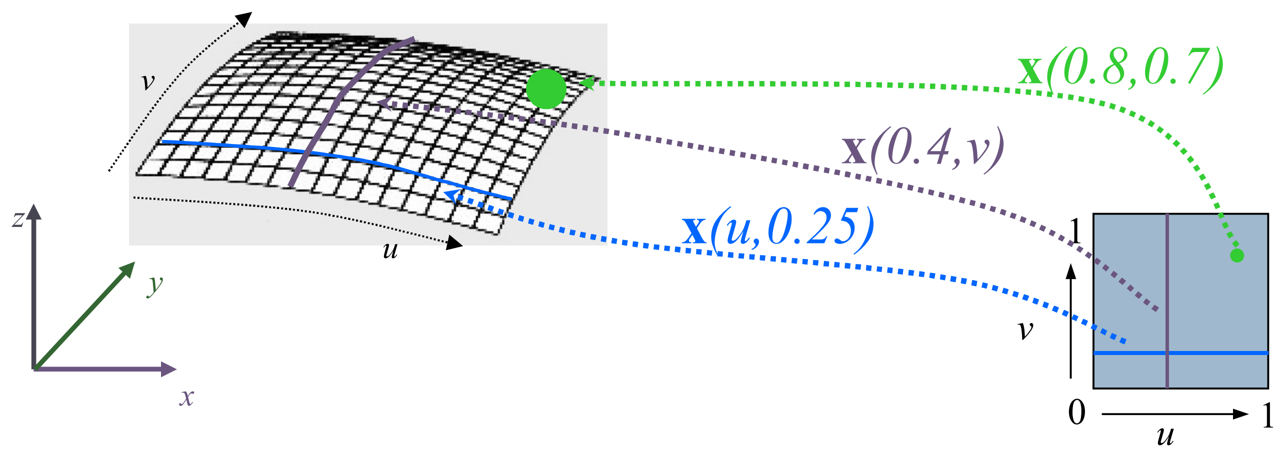
- $\mathbf{x}(u,v)$ describes a point in space for any given (u,v) pair
  - u,v each range from 0 to 1



2D parameter domain

Parametric Surface Patch

- $\mathbf{x}(u,v)$ describes a point in space for any given (u,v) pair
  - u,v each range from 0 to 1

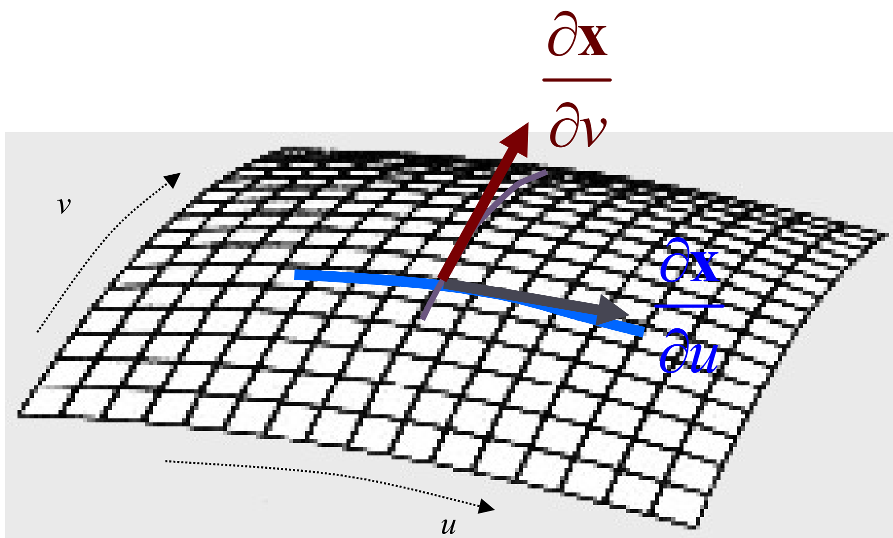


2D parameter domain

- Parametric curves
  - For fixed u_0, have a v curve $\mathbf{x}(u_0,v)$
  - For fixed v_0, have a u curve $\mathbf{x}(u,v_0)$
  - For any point on the surface, there are a pair of parametric curves through that point

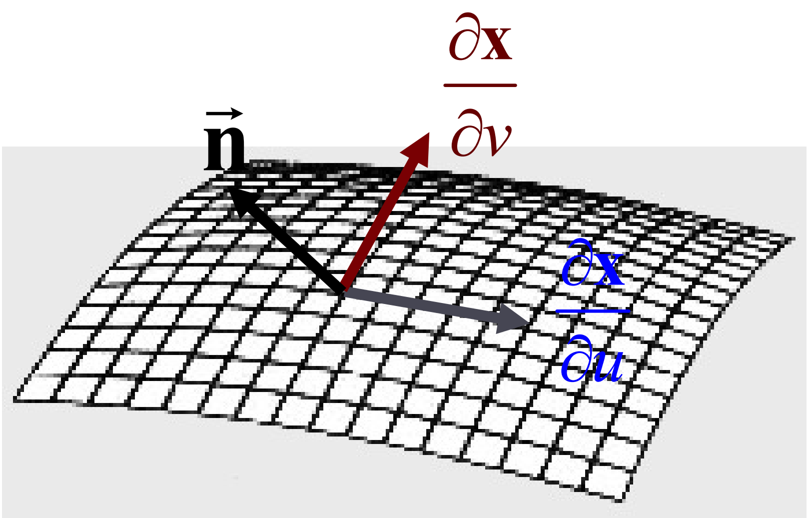
Tangents

- The tangent to a parametric curve is also tangent to the surface
- For any point on the surface, there are a pair of (parametric) tangent vectors
- Note: these vectors are not necessarily perpendicular to each other



Surface Normal

- Normal is cross product of the two tangent vectors
- Order of vectors matters!



$$\vec{\mathbf{n}}(u,v) = \frac{\partial \mathbf{x}}{\partial u}(u,v) \times \frac{\partial \mathbf{x}}{\partial v}(u,v)$$

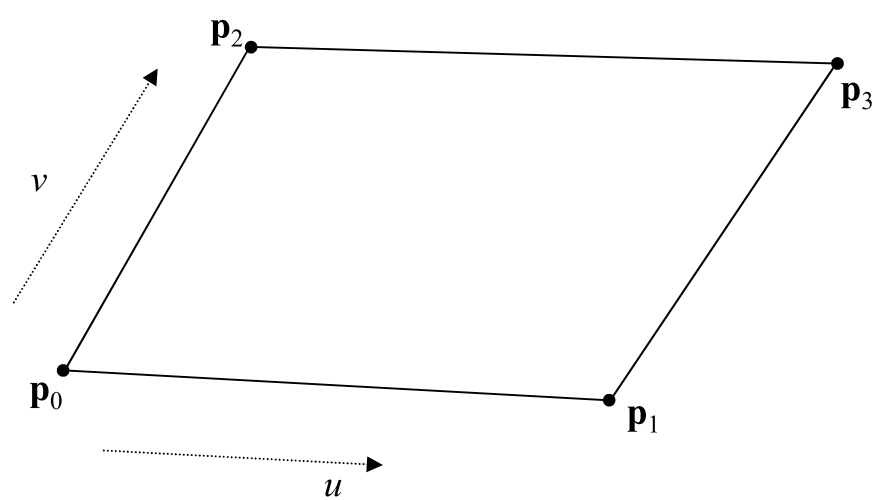
Typically we are interested in the unit normal, so we need to normalize

$$\vec{\mathbf{n}}^*(u,v) = \frac{\partial \mathbf{x}}{\partial u}(u,v) \times \frac{\partial \mathbf{x}}{\partial v}(u,v)$$

$$\vec{\mathbf{n}}(u,v) = \frac{\vec{\mathbf{n}}^*(u,v)}{\left|\vec{\mathbf{n}}^*(u,v)\right|}$$

Bilinear Patch

- Control mesh with four points $\mathbf{p_0}, \mathbf{p_1}, \mathbf{p_2}, \mathbf{p_3}$
- Compute $\mathbf{x}(u,v)$ using a two-step construction scheme

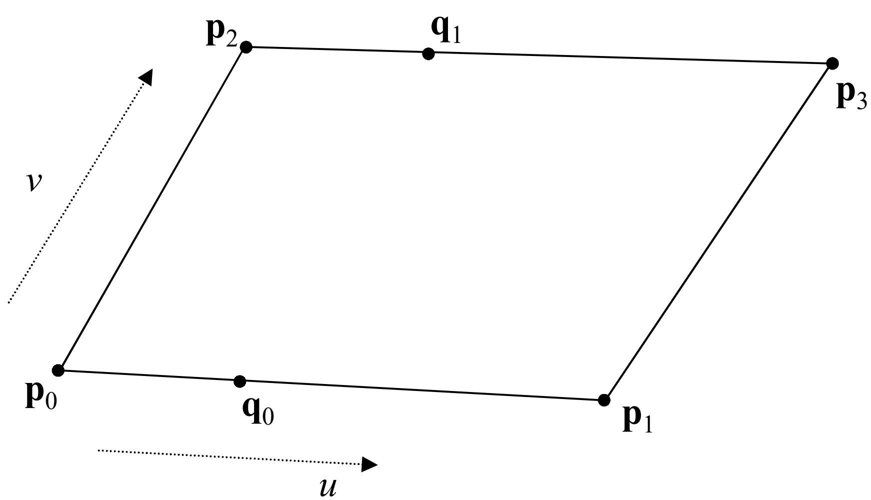


UCSD

Bilinear Patch (Step 1)

- For a given value of u, evaluate the linear curves on the two u-direction edges
- Use the same value u for both:

$$\mathbf{q_0} = Lerp(u, \mathbf{p_0}, \mathbf{p_1}) \qquad \mathbf{q_1} = Lerp(u, \mathbf{p_2}, \mathbf{p_3})$$

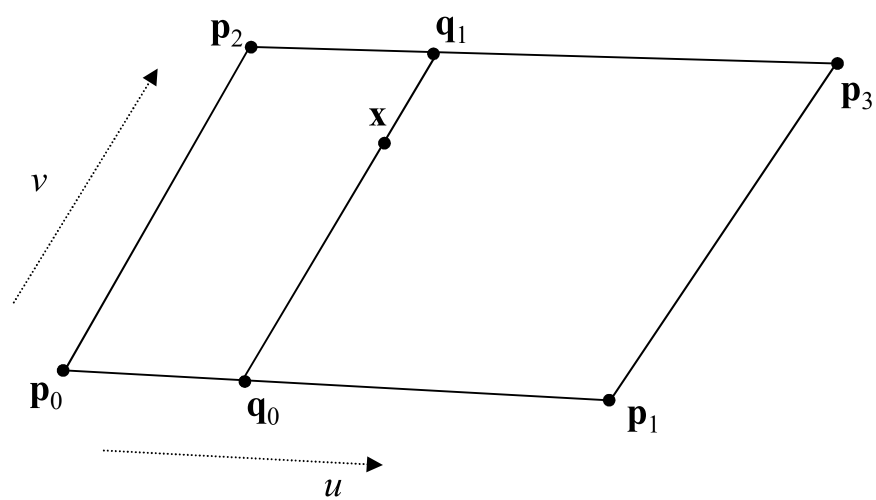


UCSD

Bilinear Patch (Step 2)

- Consider that $\mathbf{q_0}$, $\mathbf{q_1}$ define a line segment
- Evaluate it using v to get $\mathbf{x}$

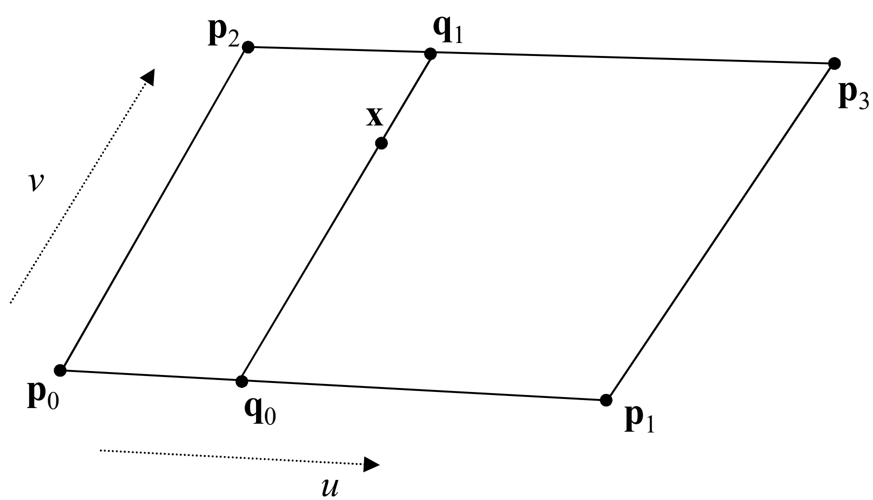
$$\mathbf{x} = Lerp(v, \mathbf{q}_0, \mathbf{q}_1)$$



Bilinear Patch

- Combining the steps, we get the full formula

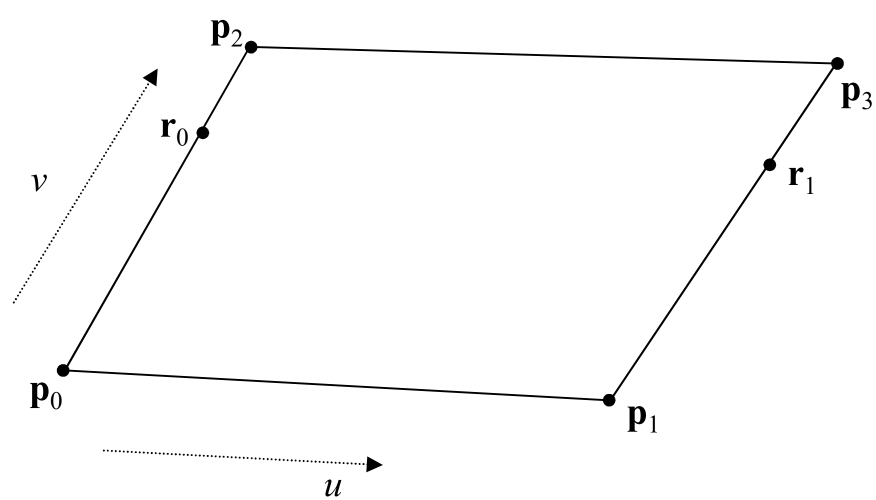
$$\mathbf{x}(u,v) = Lerp(v, Lerp(u,\mathbf{p}_0,\mathbf{p}_1), Lerp(u,\mathbf{p}_2,\mathbf{p}_3))$$



Bilinear Patch

- Try the other order
- Evaluate first in the *v* direction

$$\mathbf{r}_0 = Lerp(v, \mathbf{p}_0, \mathbf{p}_2) \qquad \mathbf{r}_1 = Lerp(v, \mathbf{p}_1, \mathbf{p}_3)$$



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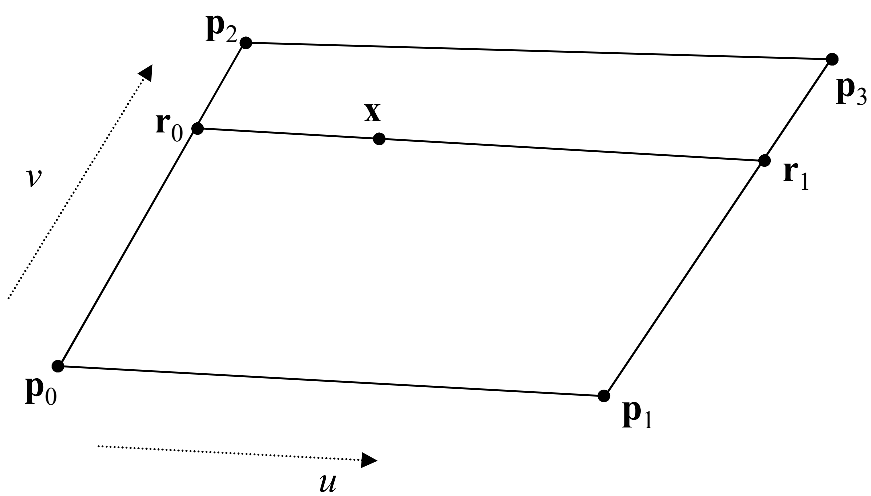
Bilinear Patch

- Consider that $\mathbf{r_0}, \mathbf{r_1}$ define a line segment
- Evaluate it using u to get $\mathbf{x}$

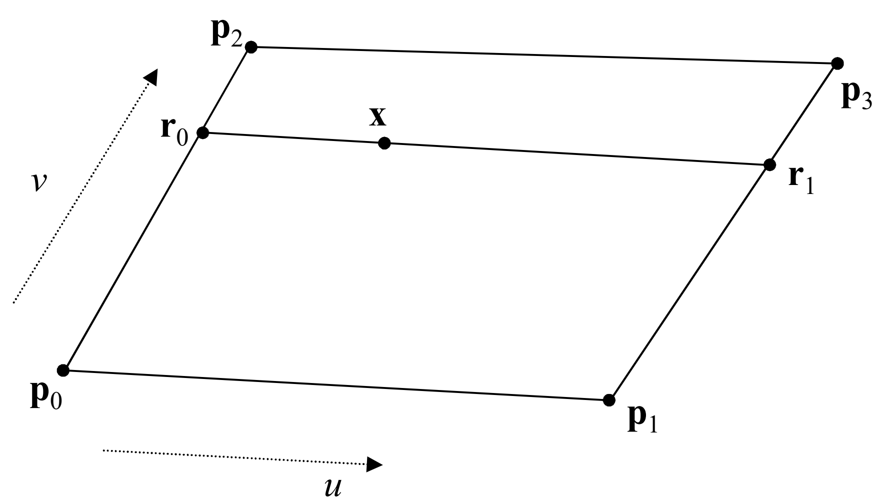
$$\mathbf{x} = Lerp(u, \mathbf{r}_0, \mathbf{r}_1)$$



Bilinear Patch

- The full formula for the v direction first:

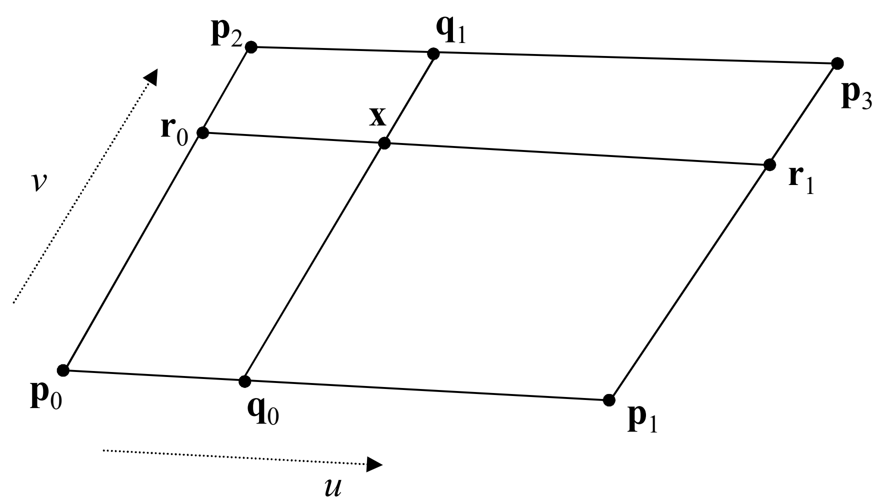
$$\mathbf{x}(u,v) = Lerp(u, Lerp(v, \mathbf{p}_0, \mathbf{p}_2), Lerp(v, \mathbf{p}_1, \mathbf{p}_3))$$



Bilinear Patch

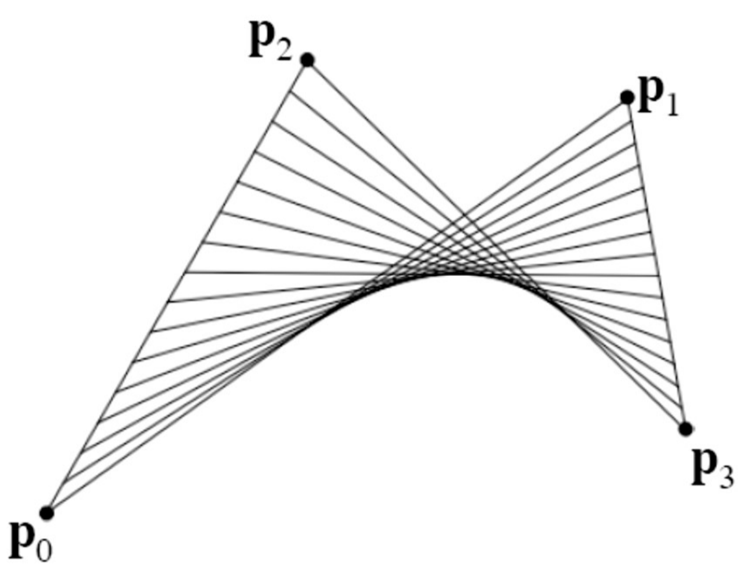
- Patch geometry is independent of the order of *u* and *v*

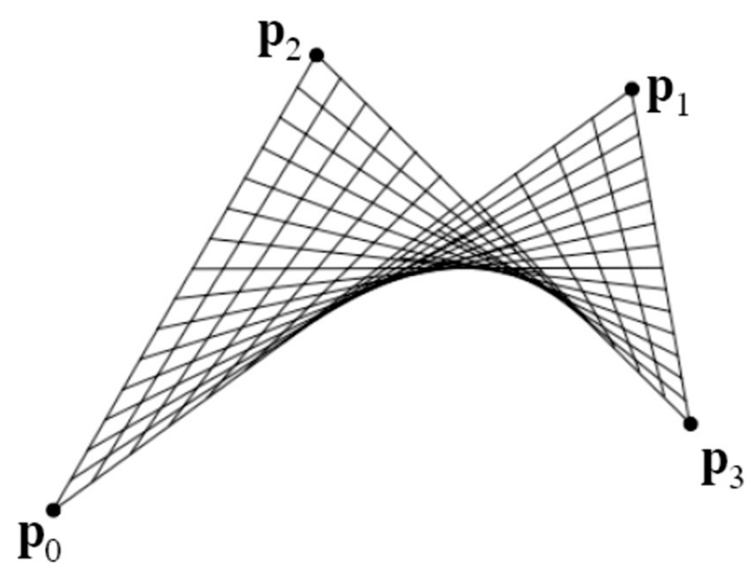
$$\boxed{\begin{aligned}\mathbf{x}(u,v) &= Lerp(v, Lerp(u,\mathbf{p}_0,\mathbf{p}_1), Lerp(u,\mathbf{p}_2,\mathbf{p}_3))\\ \mathbf{x}(u,v) &= Lerp(u, Lerp(v,\mathbf{p}_0,\mathbf{p}_2), Lerp(v,\mathbf{p}_1,\mathbf{p}_3))\end{aligned}}$$



Bilinear Patch

- Visualization





Bilinear Patches

- Weighted sum of control points

$$\mathbf{x}(u,v) = (1-u)(1-v)\mathbf{p}_0 + u(1-v)\mathbf{p}_1 + (1-u)v\mathbf{p}_2 + uv\mathbf{p}_3$$

- Bilinear polynomial

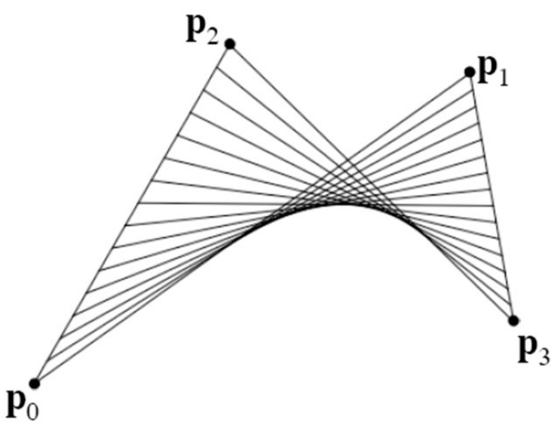
$$\mathbf{x}(u,v) = (\mathbf{p}_0 - \mathbf{p}_1 - \mathbf{p}_2 + \mathbf{p}_3)uv + (\mathbf{p}_1 - \mathbf{p}_0)u + (\mathbf{p}_2 - \mathbf{p}_0)v + \mathbf{p}_0$$

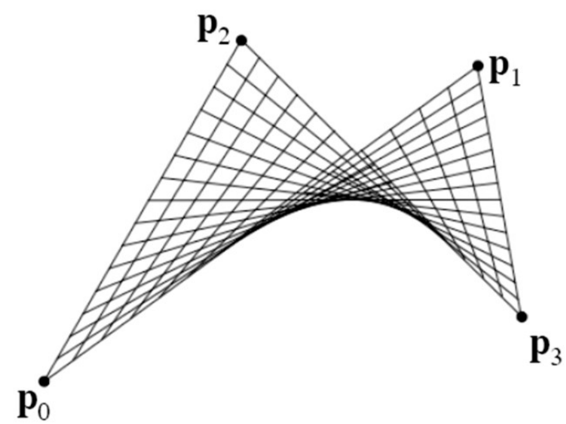
- Matrix form

$$x(u,v) = \begin{bmatrix} 1-u & u \end{bmatrix} \begin{bmatrix} p_0 & p_2 \\ p_1 & p_3 \end{bmatrix} \begin{bmatrix} 1-v \\ v \end{bmatrix}$$

Properties

- Patch interpolates the control points
- The boundaries are straight line segments
- If all 4 points of the control mesh are co-planar, the patch is flat
- If the points are not co-planar, we get a curved surface
  - saddle shape (hyperbolic paraboloid)
- *The parametric curves are all straight line segments!*
  - a (doubly) *ruled surface:* has (two) straight lines through every point





- Not terribly useful as a modeling primitive

Overview

- Bi-linear patch
- Bi-cubic Bézier patch

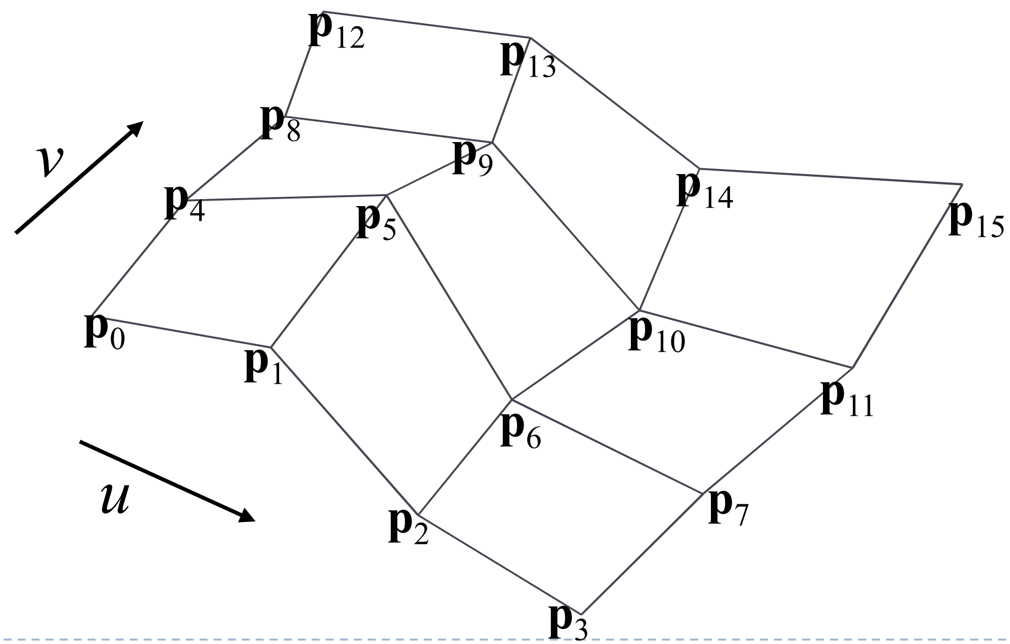
Bicubic Bézier patch

- Grid of 4x4 control points, $\mathbf{p}_0$ through $\mathbf{p}_{15}$
- Four rows of control points define Bézier curves along u

$\mathbf{p}_0,\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3;\ \ \mathbf{p}_4,\mathbf{p}_5,\mathbf{p}_6,\mathbf{p}_7;\ \mathbf{p}_8,\mathbf{p}_9,\mathbf{p}_{10},\mathbf{p}_{11};\ \mathbf{p}_{12},\mathbf{p}_{13},\mathbf{p}_{14},\mathbf{p}_{15}$

- Four columns define Bézier curves along v

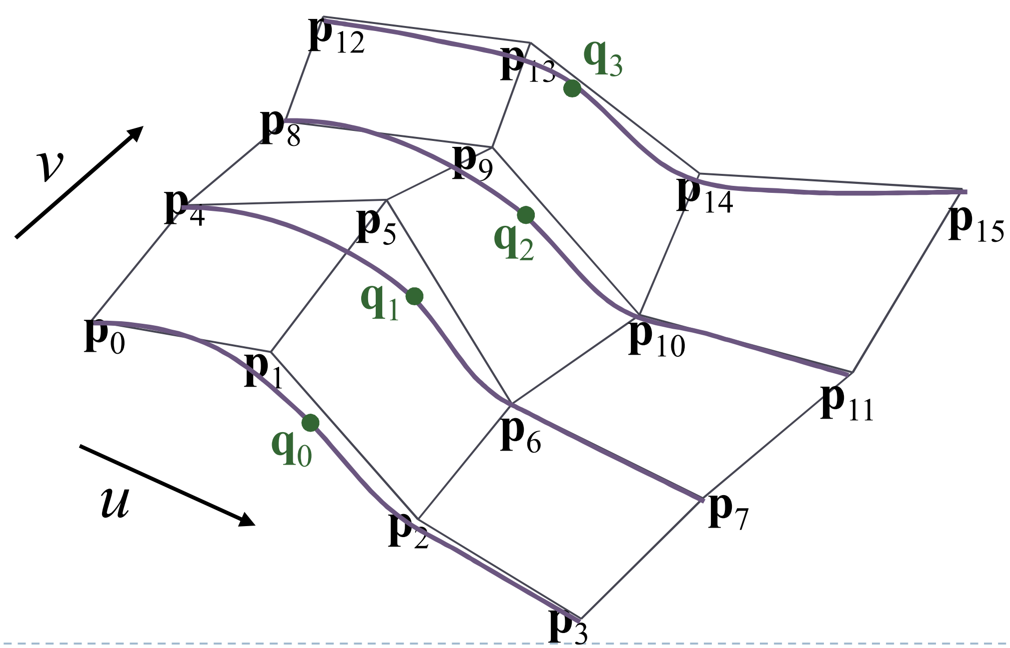
$\mathbf{p}_0,\mathbf{p}_4,\mathbf{p}_8,\mathbf{p}_{12};\ \mathbf{p}_1,\mathbf{p}_6,\mathbf{p}_9,\mathbf{p}_{13};\ \mathbf{p}_2,\mathbf{p}_6,\mathbf{p}_{10},\mathbf{p}_{14};\ \mathbf{p}_3,\mathbf{p}_7,\mathbf{p}_{11},\mathbf{p}_{15}$



Bézier Patch (Step 1)

- Evaluate four u-direction Bézier curves at scalar value u *[0..1]*
- Get points $\mathbf{q}_0 \ldots \mathbf{q}_3$

$$\mathbf{q_0} = Bez(u, \mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3)$$
$$\mathbf{q}_1 = Bez(u, \mathbf{p}_4, \mathbf{p}_5, \mathbf{p}_6, \mathbf{p}_7)$$
$$\mathbf{q}_2 = Bez(u, \mathbf{p}_8, \mathbf{p}_9, \mathbf{p}_{10}, \mathbf{p}_{11})$$
$$\mathbf{q}_3 = Bez(u, \mathbf{p}_{12}, \mathbf{p}_{13}, \mathbf{p}_{14}, \mathbf{p}_{15})$$



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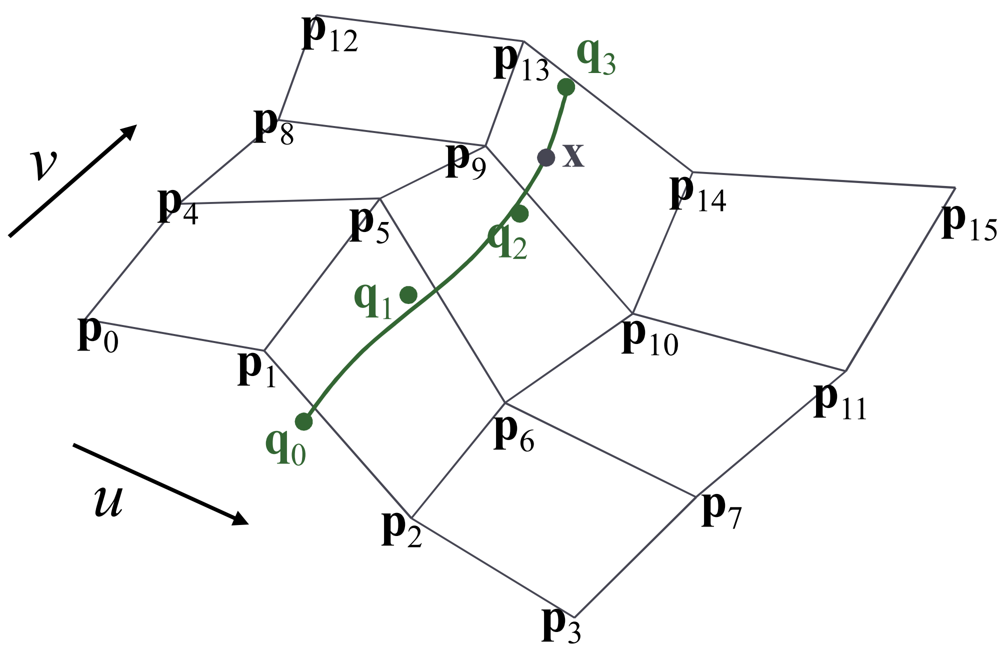
Bézier Patch (Step 2)

- Points $\mathbf{q}_0 \ldots \mathbf{q}_3$ define a Bézier curve
- Evaluate it at v *[0..1]*

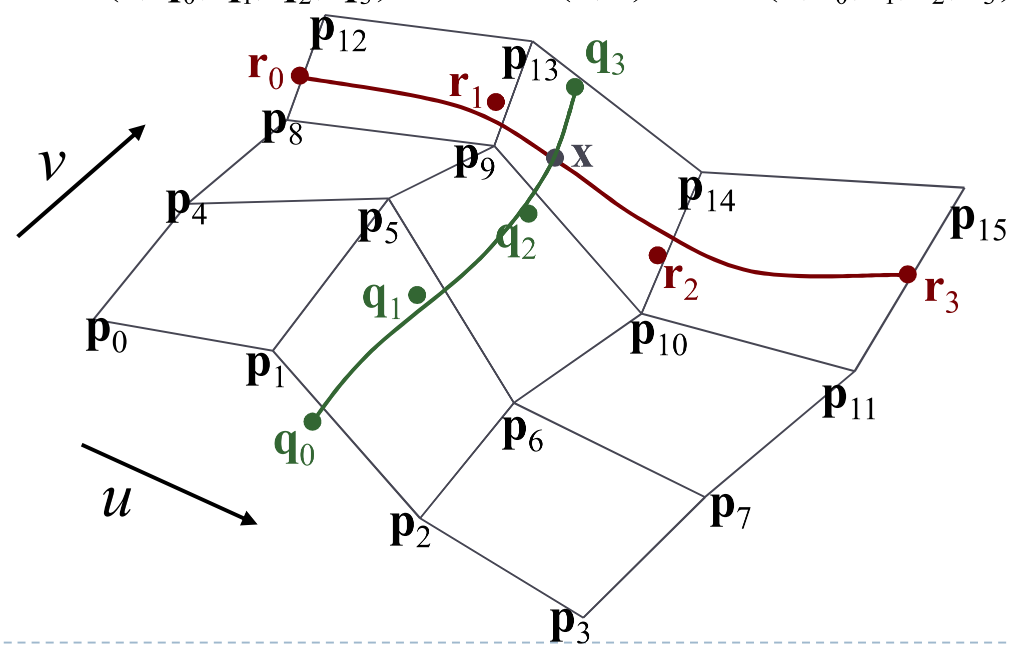
$$\mathbf{x}(u,v) = Bez(v, \mathbf{q}_0, \mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3)$$



Bézier Patch

- Same result in either order (evaluate u before v or vice versa)

$$
\begin{aligned}
\mathbf{q_0} &= Bez(u, \mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3) \\
\mathbf{q}_1 &= Bez(u, \mathbf{p}_4, \mathbf{p}_5, \mathbf{p}_6, \mathbf{p}_7) \\
\mathbf{q}_2 &= Bez(u, \mathbf{p}_8, \mathbf{p}_9, \mathbf{p}_{10}, \mathbf{p}_{11}) \\
\mathbf{q}_3 &= Bez(u, \mathbf{p}_{12}, \mathbf{p}_{13}, \mathbf{p}_{14}, \mathbf{p}_{15}) \\
\mathbf{x}(u,v) &= Bez(v, \mathbf{q}_0, \mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3)
\end{aligned}
\quad \Leftrightarrow \quad
\begin{aligned}
\mathbf{r_0} &= Bez(v, \mathbf{p}_0, \mathbf{p}_4, \mathbf{p}_8, \mathbf{p}_{12}) \\
\mathbf{r}_1 &= Bez(v, \mathbf{p}_1, \mathbf{p}_5, \mathbf{p}_9, \mathbf{p}_{13}) \\
\mathbf{r}_2 &= Bez(v, \mathbf{p}_2, \mathbf{p}_6, \mathbf{p}_{10}, \mathbf{p}_{14}) \\
\mathbf{r}_3 &= Bez(v, \mathbf{p}_3, \mathbf{p}_7, \mathbf{p}_{11}, \mathbf{p}_{15}) \\
\mathbf{x}(u,v) &= Bez(u, \mathbf{r}_0, \mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3)
\end{aligned}
$$



Bézier Patch: Matrix Form

$$\mathbf{U} = \begin{bmatrix} u^3 \\ u^2 \\ u \\ 1 \end{bmatrix} \quad \mathbf{V} = \begin{bmatrix} v^3 \\ v^2 \\ v \\ 1 \end{bmatrix} \quad \mathbf{B}_{Bez} = \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} = \mathbf{B}_{Bez}^T$$

$$\begin{aligned} \mathbf{C}_x &= \mathbf{B}_{Bez}^T \mathbf{G}_x \mathbf{B}_{Bez} \\ \mathbf{C}_y &= \mathbf{B}_{Bez}^T \mathbf{G}_y \mathbf{B}_{Bez} \\ \mathbf{C}_z &= \mathbf{B}_{Bez}^T \mathbf{G}_z \mathbf{B}_{Bez} \end{aligned}$$

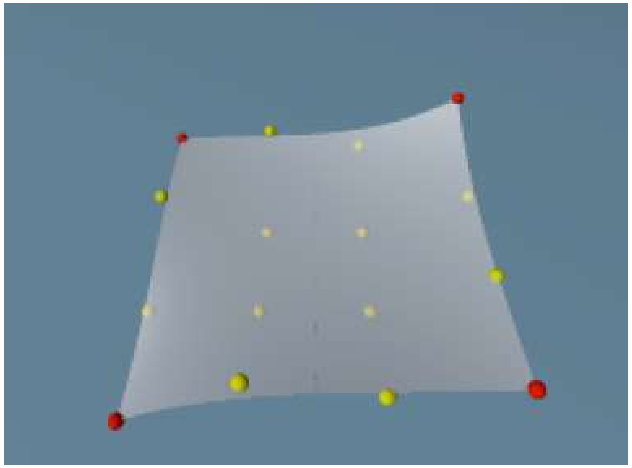
$$\mathbf{G}_x = \begin{bmatrix} p_{0x} & p_{1x} & p_{2x} & p_{3x} \\ p_{4x} & p_{5x} & p_{6x} & p_{7x} \\ p_{8x} & p_{9x} & p_{10x} & p_{11x} \\ p_{12x} & p_{13x} & p_{14x} & p_{15x} \end{bmatrix}, \ \mathbf{G}_y = \cdots, \ \mathbf{G}_z = \cdots$$

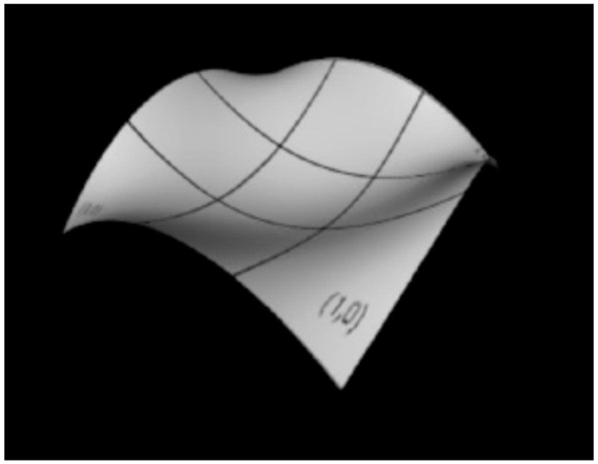
$$\boxed{\mathbf{x}(u,v) = \begin{bmatrix} \mathbf{V}^T \mathbf{C}_x \mathbf{U} \\ \mathbf{V}^T \mathbf{C}_y \mathbf{U} \\ \mathbf{V}^T \mathbf{C}_z \mathbf{U} \end{bmatrix}}$$

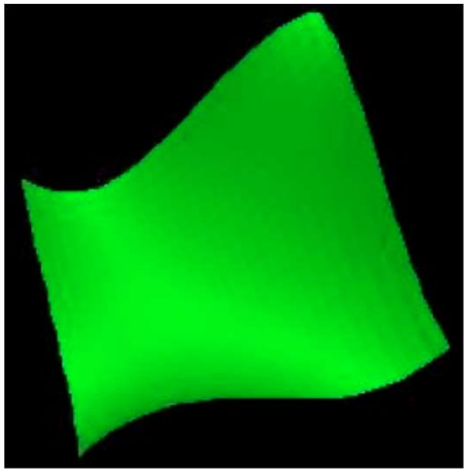
C stores the coefficients of the bicubic equation
G stores the control point geometry
$\mathbf{B}_{\text{Bez}}$ is the basis matrix (Bézier basis)
U and **V** are the vectors formed from the powers of u and v

Properties

- Convex hull: any point on the surface will fall within the convex hull of the control points
- Interpolates 4 corner points
- Approximates other 12 points, which act as "handles"
- The boundaries of the patch are the Bézier curves defined by the points on the mesh edges
- The parametric curves are all Bézier curves



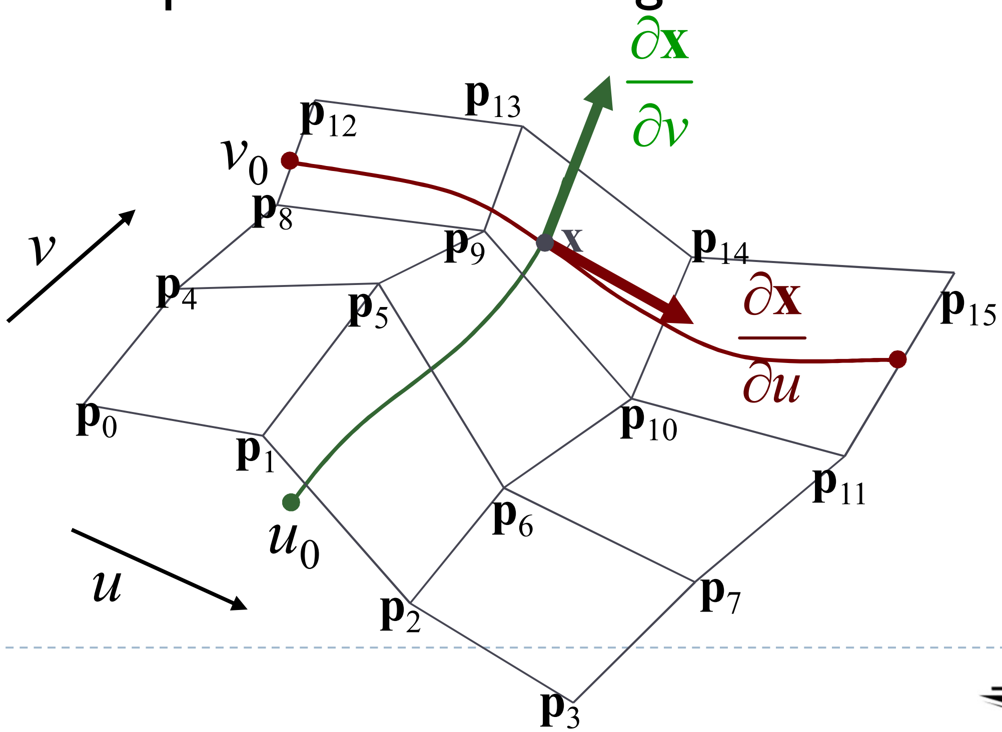




UCSD

Tangents of a Bézier patch

- Remember parametric curves $\mathbf{x}(u,v_0)$, $\mathbf{x}(u_0,v)$ where $v_{0,}\ u_0$ is fixed
- Tangents to surface = tangents to parametric curves
- Tangents are partial derivatives of $\mathbf{x}(u,v)$
- Normal is cross product of the tangents



Tangents of a Bézier patch

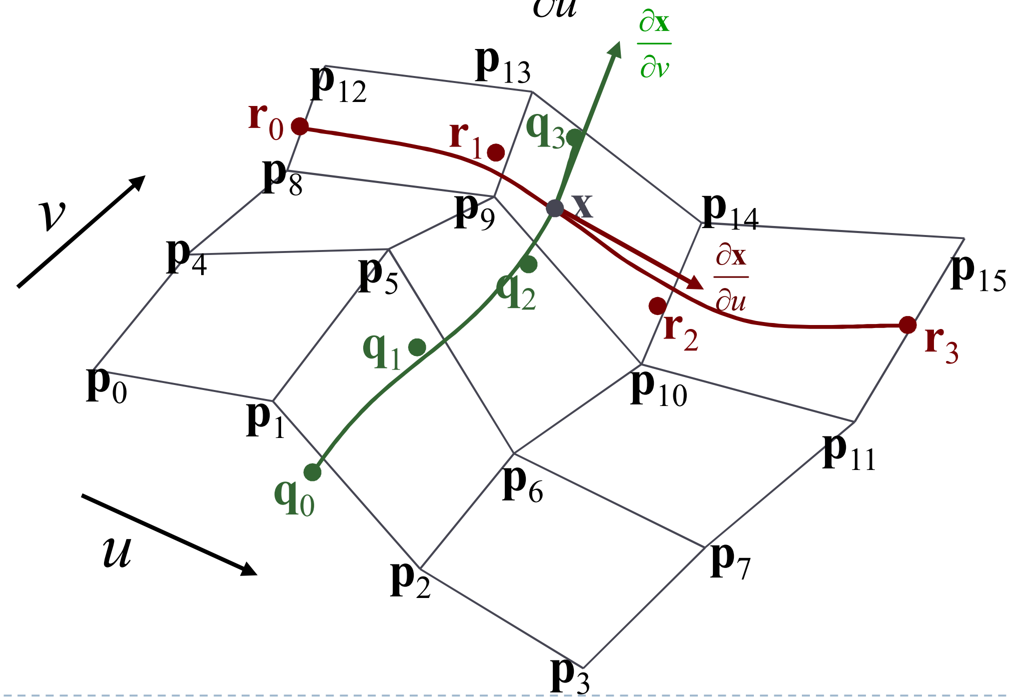
$$\mathbf{q_0} = Bez(u, \mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3) \qquad \mathbf{r_0} = Bez(v, \mathbf{p}_0, \mathbf{p}_4, \mathbf{p}_8, \mathbf{p}_{12})$$

$$\mathbf{q}_1 = Bez(u, \mathbf{p}_4, \mathbf{p}_5, \mathbf{p}_6, \mathbf{p}_7) \qquad \mathbf{r}_1 = Bez(v, \mathbf{p}_1, \mathbf{p}_5, \mathbf{p}_9, \mathbf{p}_{13})$$

$$\mathbf{q}_2 = Bez(u, \mathbf{p}_8, \mathbf{p}_9, \mathbf{p}_{10}, \mathbf{p}_{11}) \qquad \mathbf{r}_2 = Bez(v, \mathbf{p}_2, \mathbf{p}_6, \mathbf{p}_{10}, \mathbf{p}_{14})$$

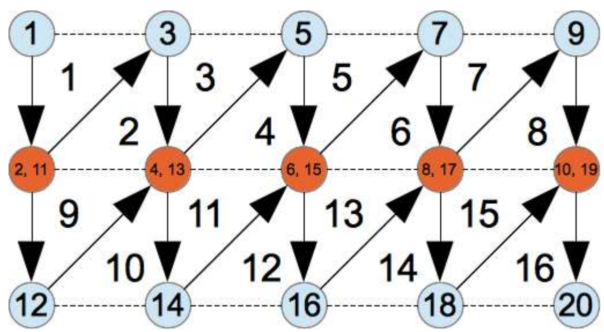
$$\mathbf{q}_3 = Bez(u, \mathbf{p}_{12}, \mathbf{p}_{13}, \mathbf{p}_{14}, \mathbf{p}_{15}) \qquad \mathbf{r}_3 = Bez(v, \mathbf{p}_3, \mathbf{p}_7, \mathbf{p}_{11}, \mathbf{p}_{15})$$

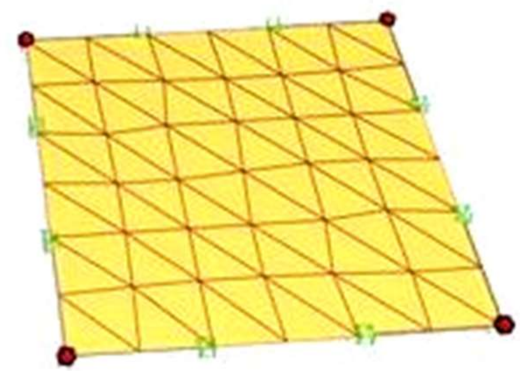
$$\frac{\partial \mathbf{x}}{\partial v}(u,v) = Bez'(v, \mathbf{q}_0, \mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3) \qquad \frac{\partial \mathbf{x}}{\partial u}(u,v) = Bez'(u, \mathbf{r}_0, \mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3)$$



Tessellating a Bézier patch

- Uniform tessellation is most straightforward
  - Evaluate points on a grid of u, v coordinates
  - Compute tangents at each point, take cross product to get per-vertex normal
  - Draw triangle strips with primitive type GL_TRIANGLE_STRIP





- Adaptive tessellation/recursive subdivision
  - Potential for "cracks" if patches on opposite sides of an edge divide differently
  - Tricky to get right, not usually worth the effort

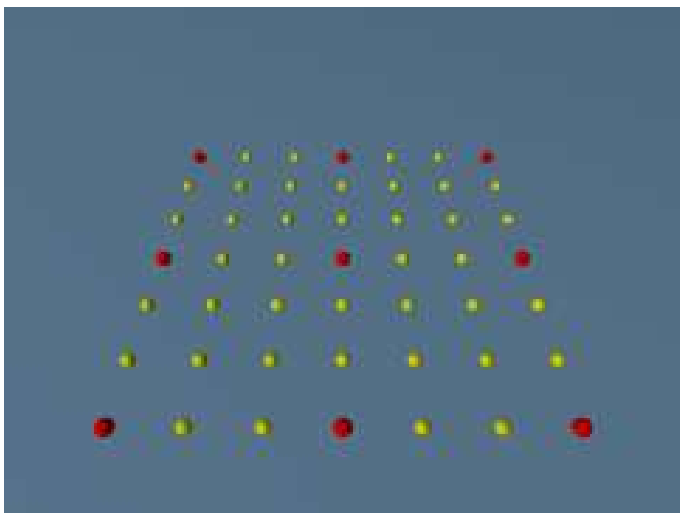
OpenGL Support

- OpenGL supports NURBS patches through GLU functions
- Structure:

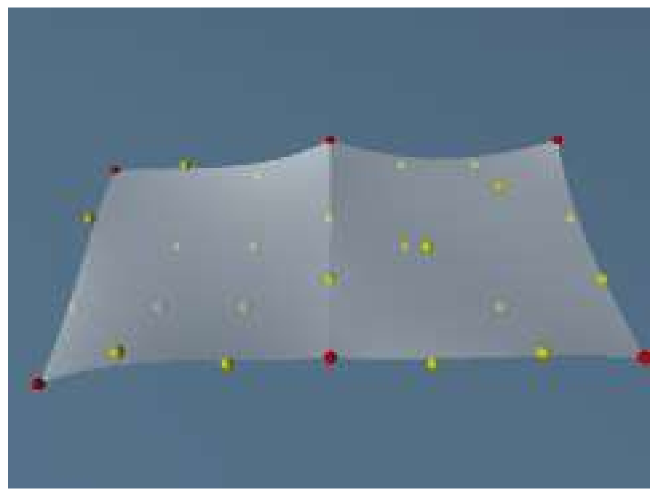
```
gluBeginSurface(nurbs);
  gluNurbsSurface(GLUnurbs* nurbs,
  GLint sKnotCount, GLfloat* sKnots,
  GLint tKnotCount, GLfloat* tKnots,
  GLint sStride, GLint tStride,
  GLfloat* control,
  GLint sOrder, GLint tOrder,
  GLenum type);
gluEndSurface(nurbs);
```

Piecewise Bézier Surface

- Lay out grid of adjacent meshes of control points
- For C^0 continuity, must share points on the edge
  - Each edge of a Bézier patch is a Bézier curve based only on the edge mesh points
  - So if adjacent meshes share edge points, the patches will line up exactly
- But we have a crease…



Grid of control points

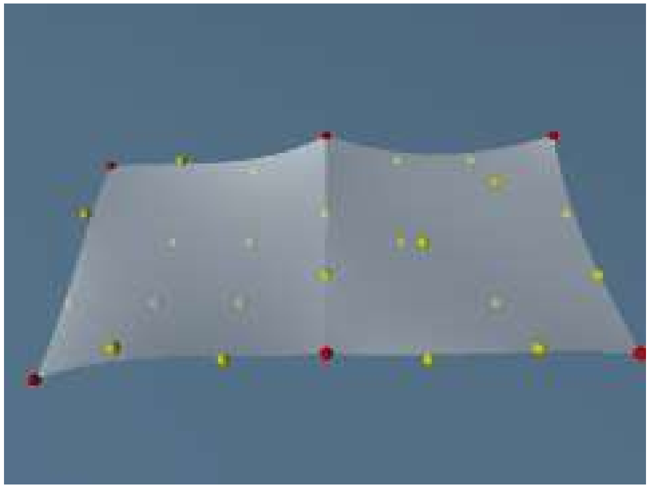


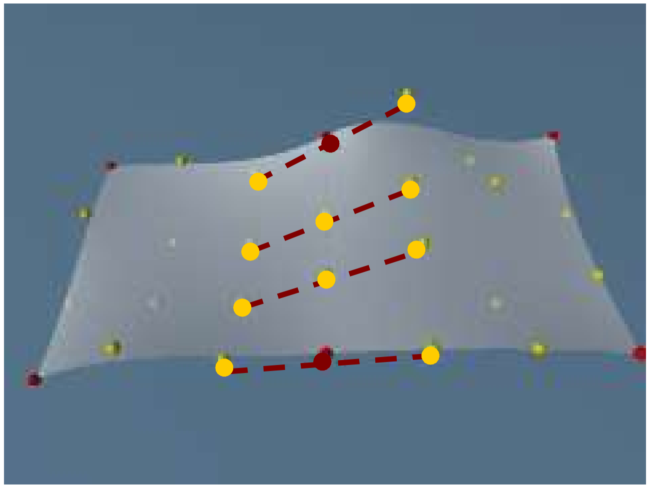
Piecewise Bézier surface

UCSD

C^1 Continuity

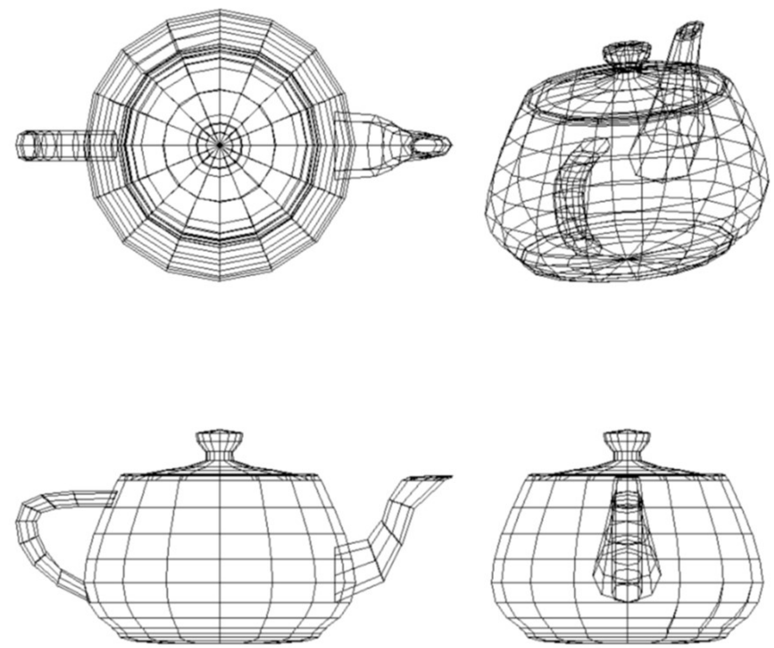
- We want the parametric curves that cross each edge to have C^1 continuity
  - So the handles must be equal-and-opposite across the edge:





Modeling With Bézier Patches

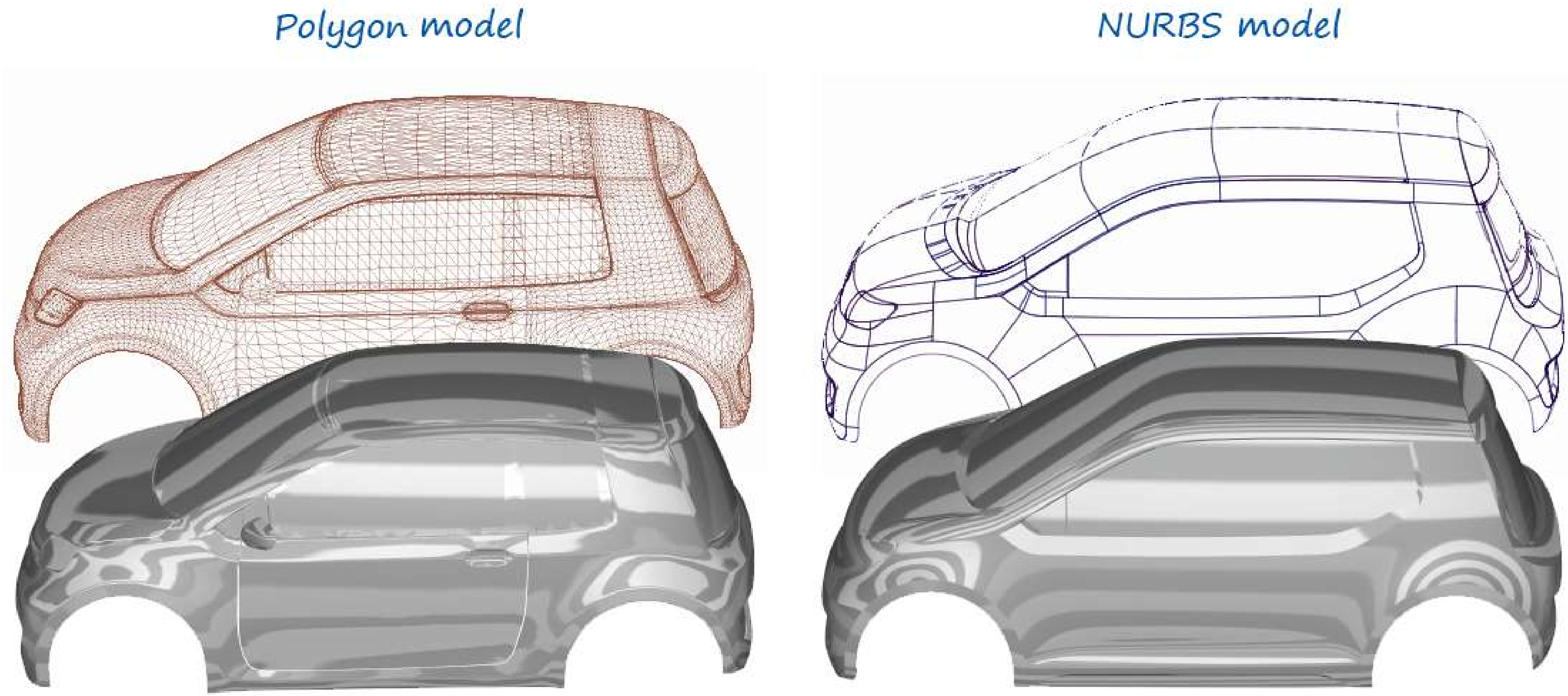
- Original Utah teapot, from Martin Newell's PhD thesis, consisted of 28 Bézier patches.
- The original had no rim for the lid and no bottom
- Later, four more patches were added to create a bottom, bringing the total to 32
- The data set was used by a number of people, including graphics guru Jim Blinn. In a demonstration of a system of his he scaled the teapot by .75, creating a stubbier teapot. He found it more pleasing to the eye, and it was this scaled version that became the highly popular dataset used today.



Pixar's walking teapot

Source: http://www.holmes3d.net/graphics/teapot/

Comparing polygon to NURBS model



Source: https://www.aliasworkbench.com